ROMANIAN MATHEMATICAL MAGAZINE

Solve for natural numbers:

$$x^3 - y^3 = 2(x^2 + y^2) + 3xy + 17$$

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$$x^3 - y^3 = 2(x^2 + y^2) + 3xy + 17 \quad (1)$$

$$Clearly, x, y \in N \; for \; this \; R.H.S \; part > 0 \text{ , } consequently \; L.H.S > 0 \text{ , } x > y$$

$$let \; d = x - y \overset{(2)}{>} 0 \; then \; x = d + y \;\; whth \; this \; substitution \; from(1) we \; get$$

$$(d + y)^3 - y^3 = 2((d + y)^2 + y^2) + 3(d + y)y + 17$$

$$d(d^2 + 2dy + y^2 + dy + y^2 + y^2) = 2(d^2 + 2dy + 2y^2) + 3y(d + y) + 17$$

$$or, y^2(3d - 7) + y(3d^2 - 7d) + (d^3 - 2d^2 - 17) = 0 \quad (2)$$

$$we \; note \; that \; x, y \in N$$

$Case - 1$:

$d = 1 \; then \; equation \; (2) becomes, -4y^2 - 4y - 18 = 0 \; or, 2y^2 + 2y + 9 = 0$, $real \; root \; does \; not \; exist \; as \; discriminant = 2^2 - 4.2.9 = -68 < 0$

$Case - 2$:

$d = 2 \; then \; equation \; (2) becomes, -y^2 - 2y - 17 = 0 \; or$ $y^2 + 2y + 17 = 0$, $real \; root \; does \; not \; exist \; as \; discriminant = 2^2 - 4.1.17 = -64 < 0$

$Case - 3$:

$$d = 3 \; then \; equation \; (2) becomes, 2y^2 + 6y - 8 = 0 \; or, (y + 4)(y - 1) = 0$$

$$or, y = 1 \; (as \; y \in N, so \; y \neq -4), x = y + d = 1 + 3 = 4$$

$$now \; from(2) \; for \; d \geq 4 \; coeffcient \; of \; y^2 \geq 3 \times 4 - 7 = 5 > 0$$

$$coeffcient \; of \; y \geq 3 \times 4^2 - 7 \times 4 = 20 > 0$$

$$constant \; term \geq 4^3 - 32 - 17 = 64 - 49 = 15 > 0$$

$$clearly \; for \; d \geq 4 \; equation \; (2) \; never \; be \;\; zero.$$

$$Required \; solution \; in \; N \; is \; x = 4, y = 1$$