ROMANIAN MATHEMATICAL MAGAZINE

If $\lambda \in \mathbb{N}^*$ fixed then solve for natural numbers:

$$\frac{(x+y+z)^5 - x^5 - y^5 - z^5}{(x+y+z)^3 - x^3 - y^3 - z^3} = 10\lambda^2$$

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Solution by Tapas Das-India

$$(x+y+z)^5 = x^5 + y^5 + z^5 + 5(x+y)(y+z)(z+x)(x^2+y^2+z^2+xy+yz+zx)$$

$$(x+y+z)^5 - x^5 - y^5 - z^5 =$$

$$= 5(x+y)(y+z)(z+x)(x^2+y^2+z^2+xy+yz+zx)(1)$$

$$(x+y+z)^3 = x^3 + y^3 + z^3 + 3(x+y)(y+z)(z+x)$$

$$(x+y+z)^3 - x^3 - y^3 - z^3 = 3(x+y)(y+z)(z+x)\ (2)$$

$$\frac{(x+y+z)^5 - x^5 - y^5 - z^5}{(x+y+z)^3 - x^3 - y^3 - z^3} = 10\lambda^2$$

$$\frac{5}{3}(x^2+y^2+z^2+xy+yz+zx) \overset{(1)\&(2)}{=} 10\lambda^2(3)$$

$$\textit{The equation is symmetric in } x, y, z \textit{ and' homogeneus of degree } 2$$

$$\textit{hence we examine symmetric case } x = y = z = t(\textit{say})$$

$$\textit{From } (3) \textit{we get } 6t^2 = 6\lambda^2 \textit{ or } t = \lambda$$

$$\textit{Solution } x = y = z = \lambda$$