ROMANIAN MATHEMATICAL MAGAZINE

If $\lambda \in \mathbb{Z}$ fixed then prove that the equation:

$$x^3 + y^3 - z^3 = \lambda^2(x + y - z)$$

has an infinitely solutions in integers.

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$$We\ are\ asked\ to\ prove\ existence\ of\ infinitely\ many\ integer\ solution,$$

$$not\ to\ find\ all\ solution, so\ it\ is\ enough\ to\ find\ of\ solution\ one\ infinite\ family$$

$$for\ this\ we\ put\ x + y - z = 0 \Rightarrow z = x + y\ then\ the\ equation\ becomes$$

$$x^3 + y^3 - (x + y)^3 = 0\ or - 3xy(x + y) = 0\ \ \Rightarrow\ x = 0\ or\ y = 0\ or\ x + y = 0$$

$$Case\ 1)\ x = 0\ then\ z = y\ as\ z = x + y\ , solution\ (0, y, y), y \in z$$

$$Case\ 2)\ y = 0\ then\ z = x\ as\ z = x\ , solution\ (x, 0, x), x \in z$$

$$Case\ 3) x + y = 0\ then\ y = -x, z = 0\ as\ z = x + y\ , solution\ (x, -x, 0), x \in \mathbb{Z}$$

$$For\ all\ these\ solutions\ x + y - z = 0 \Rightarrow\ \lambda^2(x + y - z) = 0\ then\ x^3 + y^3 - z^3 = 0$$

$$then\ orginal\ equation\ is\ satisfied\ for\ every\ integer\ \lambda\ , (0, y, y), (x, 0, x), (x, -x, 0)$$

$$x, y, z \in\ Z\ infinitely\ many\ integer\ solutions\ .$$