ROMANIAN MATHEMATICAL MAGAZINE

Solve for integers:

$$2x^2 + 6y^2 + 7xy - x - y = 4$$

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Let $x, y \in \mathbb{Z}$. Factoring the quadratic part of the equation:

$$2x^2 + 7xy + 6y^2 = (2x + 3y)(x + 2y)$$

By rearranging the terms: $(2x + 3y)(x + 2y) - x - y = 4$

Subtracting 1 from both sides: $(2x + 3y)(x + 2y) - (x + y) - 1 = 3$

Therefore: $(2x + 3y + 1)(x + 2y - 1) = 3$

Since $x, y \in \mathbb{Z}$, the terms $(2x + 3y + 1)$ and $(x + 2y - 1)$ must be integer divisors of 3:

$$(2x + 3y + 1, x + 2y - 1) \in \{(1, 3), (3, 1), (-1, -3), (-3, -1)\}$$

Solving the corresponding linear systems:

1. $\begin{cases} 2x + 3y + 1 = 1 \\ x + 2y - 1 = 3 \end{cases} \Rightarrow \begin{cases} 2x + 3y = 0 \\ x + 2y = 4 \end{cases} \Rightarrow (x, y) = (-12, 8)$
2. $\begin{cases} 2x + 3y + 1 = 3 \\ x + 2y - 1 = 1 \end{cases} \Rightarrow \begin{cases} 2x + 3y = 2 \\ x + 2y = 2 \end{cases} \Rightarrow (x, y) = (-2, 2)$
3. $\begin{cases} 2x + 3y + 1 = -1 \\ x + 2y - 1 = -3 \end{cases} \Rightarrow \begin{cases} 2x + 3y = -2 \\ x + 2y = -2 \end{cases} \Rightarrow (x, y) = (2, -2)$
4. $\begin{cases} 2x + 3y + 1 = -3 \\ x + 2y - 1 = -1 \end{cases} \Rightarrow \begin{cases} 2x + 3y = -4 \\ x + 2y = 0 \end{cases} \Rightarrow (x, y) = (-8, 4)$

Hence, the set of integer solutions are: $(x, y) \in \{(-12, 8), (-2, 2), (2, -2), (-8, 4)\}$