ROMANIAN MATHEMATICAL MAGAZINE

Solve for integers:

$$\lfloor\sqrt[3]{1}\rfloor+\lfloor\sqrt[3]{2}\rfloor+\lfloor\sqrt[3]{3}\rfloor+\cdots+\lfloor\sqrt[3]{x^3-1}\rfloor=400$$

$$where\ \lfloor*\rfloor-\ floor\ function$$

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$$\lfloor\sqrt[3]{1}\rfloor+\lfloor\sqrt[3]{2}\rfloor+\lfloor\sqrt[3]{3}\rfloor+\cdots+\lfloor\sqrt[3]{x^3-1}\rfloor=400\to\sum_{k=1}^{x^3-1}\lfloor\sqrt[3]{k}\rfloor=400$$

$$Floor\ function\ \lfloor\sqrt[3]{k}\rfloor=n\to n\le\sqrt[3]{k}<n+1\to n^3\le k<(n+1)^3$$

$$k\in[n^3;(n+1)^3)\to k\in[n^3;(n+1)^3]$$

The number of integers in this interval $P_n=(n+1)^3-n^3=3n^2+3n+1$

$\sqrt[3]{x^3-1}<x\to\lfloor\sqrt[3]{x^3-1}\rfloor=x-1.$ Thus, n ranges from 1 to $x-1$

$$\sum_{n=1}^{x-1}n\cdot P_n=400\to\sum_{n=1}^{x-1}n\cdot(3n^2+3n+1)=400$$

$$3\sum_{n=1}^{x-1}n^3+3\sum_{n=1}^{x-1}n^2+\sum_{n=1}^{x-1}n=400$$

$$3\cdot\frac{(x-1)^2x^2}{4}+3\cdot\frac{x(x-1)(2x-1)}{6}+\frac{x(x-1)}{2}=400$$

$$(x-1)\left(\frac{3x^3-3x^2}{4}+\frac{2x^2-x}{2}+\frac{x}{2}\right)=400\to(x-1)(3x^3+x^2)=1600$$

$$3x^4-2x^3-x^2-1600=0$$

$$3x^4-15x^3+13x^3-65x^2+64x^2-1600=0$$

$$3x^3(x-5)+13x^2(x-5)+64(x-5)(x+5)=0$$

$$(x-5)(3x^3+13x^2+64x+320)=0\to\boxed{x_1=5}$$

$3x^3+13x^2+64x+320\ne0$ *as* $x\in N$, *There is no other natural solution.*