ROMANIAN MATHEMATICAL MAGAZINE

$$(1+x+x^2+x^3+\cdots+x^{100})^3=1+\cdots+kx^{100}+\cdots+x^{300}$$

$$k=?$$

Proposed by Mais Hasanov-Azerbaijan

Solution by Amin Hajiyev-Azerbaijan

$$f(x)=\Big(\underbrace{1+x+x^2+\cdots+x^{100}}_{101}\Big)$$

$$g(x)=\Big(\underbrace{1+x+x^2+x^3+\cdots+x^{100}+\cdots}_{\infty}\Big)$$

$$f(x)(coefficient\ x^{100})=g(x)(coefficient\ x^{100})$$

$$g(x)=(1+x+x^2+\cdots)^3=\left(\sum_{n=0}^{\infty}x^n\right)^3=\left(\frac{1}{1-x}\right)^3$$

$$Note:\ p(x)=(1+x+x^2+\cdots)^n=\left(\frac{1}{1-x}\right)^n=\sum_{m=0}^{\infty}\binom{m+n-1}{m}x^m$$

$$g(x)=\sum_{m=0}^{\infty}\binom{m-1+3}{m}x^m=\sum_{m=0}^{\infty}\binom{m+2}{m}x^m$$

$$m=100\rightarrow\ g(x)(coefficient\ x^{100})=x^{100}\binom{102}{100}=\frac{102!}{100!\,2!}x^{100}=5151x^{100}$$

$$k=5151$$