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Find all pairs of positive integers (x, y) satisfying

$$25^x + (4^y + 1)(4y^2 + 3y + 3) = 5^x(4^y + 4y^2 + 3y + 4)$$

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Suppose, $(4^y + 1) = k$ and $(4y^2 + 3y + 3) = w$

Now, the equation becomes: $5^{2x} + kw = 5^x(k + w) \Rightarrow 5^{2x} - 5^x(k + w) + kw = 0$

After solving the quadratic equation, we get

Case 1 $5^x = 4^y + 1$

If $x = 0$, $5^0 = 4^y + 1 \Rightarrow 1 - 1 = 4^y \Rightarrow y$ has no integer solution.

If $x = 1$, then, $5^1 = 4^y + 1 \Rightarrow y = 1$. So, we need to find the integer pairs which are $x \geq 1$ and $y \geq 1$, $4^y + 1 \equiv 1 \pmod{16}$ if $y > 1$, $5^2 \equiv 9 \pmod{16}$,

$$5^3 \equiv 13 \pmod{16}, \quad 5^4 \equiv 1 \pmod{16}$$

So, $4 \mid x \Rightarrow x = 4m \quad m \in \mathbb{Z}^+$. So, $4^y = 5^x - 1 \Rightarrow 4^y = 5^{4k} - 1 = (5^{2k} - 1)(5^{2k} + 1)$

Assume that $5^{2k} - 1 = A$ and $5^{2k} + 1 = B$, so, $\gcd(A, B) = 2$ as A and B are both even.

So, $A = 2u$ and $B = 2v \quad u, v \in \mathbb{Z}^+, \gcd(u, v) = 1$

If, $4^y = A \cdot B \Rightarrow 2^{2y} = 4uv \Rightarrow 2^{2y-2} = uv$. If $u = 1$ then, $A = 2$ So, $5^{2k} = 3$ a contradiction.

Hence, no solution exist for $y \geq 2$, So, Case 1 has one solution $(x, y) = 1, 1$.

Case 2 $5^x = 4y^2 + 3y + 3$ (as $w = 4y^2 + 3y + 3 \Rightarrow 4y^2 + 3y + 3 \equiv y^2 \pmod{3}$)

$$5^x \equiv 2^x \pmod{3}$$

If x is even $2^x \not\equiv 2 \pmod{3}$

Let $2 \mid x \Rightarrow x = 2n \quad n \in \mathbb{Z}^+$ then, $5^x = (5^n)^2$ is a perfect square. But for $y > 2$

$$(2y)^2 < 4y^2 + 3y + 3 < 4y^2 + 4y + 1 = (2y + 1)^2$$

So, no perfect square cannot lie between $(2y)^2$ and $(2y + 1)^2$

because, $3y + 3 < 4y + 1 \Rightarrow y > 2$

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So, no solution with even $x \geq 2$ and $y > 2$

So we need to find $0 < y \leq 2$

If $y = 1$; $4 + 3 + 3 = 10 \neq 5^x$

If $y = 2$; $16 + 6 + 3 = 25 = 5^2 = 5^x$, $x = 2$

So, Case 2 have $x = 2, y = 2$ as one solution.

Therefore, the possible ordered pairs of positive integers $(x, y) = (1, 1), (2, 2)$