ROMANIAN MATHEMATICAL MAGAZINE

Solve for integers:

$$\begin{cases}(x+y+z)^2+(x-y+z)^2-18(z-1)(5-z)=2y^2\\ x+y-z=4\end{cases}$$

Proposed by Bui Hong Suc-Vietnam

Solution by Tapas Das-India

$$(x+y+z)^2+(x-y+z)^2-18(z-1)(5-z)=2y^2$$
$$or, 2((x+z)^2-y^2)-18(z-1)(5-z)=2y^2$$
$$or, (x+z)^2=9(z-1)(5-z)$$
$$or, (4+2z-y)^2\overset{x+y-z=4}{=}9(z-1)(5-z)\ \ (1)$$

$$now\ L.H.S\geq 0\ so\ R.H.S\geq 0\ for\ this\ (z-1)(5-z)\geq 0\ or, 1\leq z\leq 5$$

$when\ z=1, R.H.S=0\ so\ from\ (1) we\ get\ (6-y)^2=0\ or, y=6$
$and\ x=4+z-y=4+1-6=-1\ , solution\ (x,y,z)=(-1,6,1)$

$when\ z=2, R.H.S=27\ so\ from\ (1) we\ get\ (8-y)^2=27$
$no\ integer\ solution\ exist$

$when\ z=3, R.H.S=36\ so\ from\ (1) we\ get\ (10-y)^2=36\ or, y=4,16$
$and\ x=4+z-y\overset{y=4}{=}4+3-4=3\ , solution\ (x,y,z)=(3,4,3)$
$and\ x=4+z-y\overset{y=16}{=}4+3-16=-9\ , solution\ (x,y,z)=(-9,16,3)$

$when\ z=4, R.H.S=27\ so\ from\ (1) we\ get\ (12-y)^2=27$
$no\ integer\ solution\ exist$

$when\ z=5, R.H.S=0\ so\ from\ (1) we\ get\ (14-y)^2=0\ or, y=14$
$and\ x=4+z-y=4+5-14=-5$
$solution\ (x,y,z)=(-5,14,5)$