ROMANIAN MATHEMATICAL MAGAZINE

Find $x, y, z \in Z$ such that:

$$x^4 + y^4 = 21 - x^2y^2$$

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$$x^4 + y^4 = 21 - x^2y^2 \ \ or, \qquad x^4 + y^4 + x^2y^2 = 21$$

$$or, a^2 + b^2 + ab \overset{a=x^2\geq 0, b=y^2\geq 0}{=} 21 \ (1)$$

$for\ a = 0\ \ from\ (1) we\ get\ b^2 = 21 \neq Z\ no\ integer\ solution$

$for\ a = 1\ from\ (1) we\ get\ 1 + b^2 + b = 21\ or, (b - 4)(b + 5) = 0\ or, b = 4$

$$(as\ b = -5 \neq Z)$$

$for\ a = 2\ from\ (1) we\ get\ 4 + b^2 + 2b = 21\ or, b^2 + 2b - 17 = 0$

$$Discriminant = D = 2^2 - 4.1.(-17) = 72\ not\ a\ perfect\ square\ ,$$

$$no\ integer\ solution$$

$for\ a = 3\ from\ (1) we\ get\ 9 + b^2 + 3b = 21\ or, b^2 + 3b - 12 = 0$

$$Discriminant = D = 3^2 - 4.1.(-12) = 57\ not\ a\ perfect\ square\ ,$$

$$no\ integer\ solution$$

$for\ a = 4\ from\ (1) we\ get\ b^2 + 4b - 5 = 0\ \ or, (b - 1)(b + 5) = 0\ or, b = 1$

$$(as\ b = -5 \neq Z)$$

$$for\ a \geq 5\ L.H.S\ of\ (1) \geq 21\ not\ possible$$

$$now\ a = 1, b = 4 \Rightarrow\ x^2 = 1, y^2 = 4\ or, x = \pm 1, y = \pm 2$$

$$solution: (1,2), (1,-2), (-1,2), (-1,-2)$$

$$now\ a = 4, b = 1 \Rightarrow\ x^2 = 4, y^2 = 1\ or, x = \pm 2, y = \pm 1$$

$$solution: (2,1), (2,-1), (-2,1), (-2,-1)$$