

ROMANIAN MATHEMATICAL MAGAZINE

Find all pairs of integers (x, y) satisfying:

$$x^4 + 4x + 1 = 5^y$$

Proposed by Bibek Yadav-Nepal

Solution by Jenish Rijal-Bardiya-Nepal

If $y < 0$ then the equation has no solution because 5^y is not an integer.

For $y \geq 0$, 5^y is odd, hence $x^4 + 4x$ must be even $\Rightarrow x$ is even.

Here, $x^4 + 4x \equiv 0 \pmod{8} \Rightarrow x^4 + 4x + 1 \equiv 1 \pmod{8} \Rightarrow 5^y \equiv 1 \pmod{8}$

$\Rightarrow y$ is even.

If $x = 0$, then $y = 0$. If $x = 2$, then $y = 2$. Also if $x = -2$, then $5^y = 9$ (not possible.)

Now, if $|x| \geq 4$, then, we have,

$$x^4 + 4x + 1 = 5^y$$

which is a perfect square (as y is even). But

- $\triangleright x^4 + 4x + 1 - (x^2 - 1)^2 = 2x(x + 2) > 0$
- $\triangleright (x^2 + 1)^2 - (x^4 + 4x + 1) = 2x(x - 2) > 0$

So, $(x^2 - 1)^2 < x^4 + 4x + 1 < (x^2 + 1)^2$ for $|x| \geq 4$

The only possibility now is: $x^4 + 4x + 1 = (x^2)^2 = x^4$, which is not possible.

Therefore, the only possible solutions are $(x, y) = (0, 0)$ & $(2, 2)$.