ROMANIAN MATHEMATICAL MAGAZINE

Let $\langle a_n\rangle_{n\geq 1}$ *and* $\langle b_n\rangle_{n\geq 0}$ *be two sequences defined by*

- $a_1 = 1$ *and* $a_n = a_{n-1} + 2n - 1, (n \geq 2)$
- $b_0 = 2, b_1 = 3$ *and* $b_n = 3b_{n-1} - 2b_{n-2}, (n \geq 2)$

Determine all positive integers n *for which* a_n *divides* b_n.

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$$Here, a_n - a_{n-1} = 2n - 1 \Rightarrow \sum_{m=2}^{n}(a_m - a_{m-1}) = \sum_{m=2}^{n}(2m-1) \Rightarrow a_n - a_1$$
$$= n^2 - 1 \quad \therefore a_n = n^2$$

$$Similarly,\ b_n - b_{n-1} = 2b_{n-1} - 2b_{n-2} \Rightarrow \frac{b_n - b_{n-1}}{b_{n-1} - b_{n-2}} = 2$$

$$\Rightarrow \prod_{m=2}^{n}\left(\frac{b_m - b_{m-1}}{b_{m-1} - b_{m-2}}\right) = \prod_{m=2}^{n}(2)$$

$$\Rightarrow \frac{b_n - b_{n-1}}{b_1 - b_0} = 2^{n-1} \Rightarrow \sum_{m=1}^{n}(b_m - b_{m-1}) = \sum_{m=1}^{n}(2^{m-1}) \Rightarrow b_n - b_0 = 2^n - 1$$
$$\therefore b_n = 2^n + 1$$

$$Thus,\quad a_n \text{ divides } b_n \Leftrightarrow n^2 \mid 2^n + 1$$

Let's solve for $n^2 \mid 2^n + 1$: *Trivially* $n = 1$ *is a solution.*

If $n > 1$ *then define* $\omega(t)$ *to be the smallest prime divisor of* $t \in \mathbb{N}$.

Let $\omega(n) = p$. $\quad \Rightarrow p \mid 2^n + 1 \mid 2^{2n} - 1$ *and* $p \mid 2^{p-1} - 1$ (*By FLT*)

$$\Rightarrow p \mid gcd(2^{2n} - 1, 2^{p-1} - 1) \Rightarrow p \mid 2^{gcd(2n,p-1)} - 1$$

Now if $d = gcd(2n, p-1) > 2$

then $\frac{d}{2}$ *is a divisor of* n *strictly less than* p, *implying*

n *has a prime factor smaller than* p, *which contradicts the minimality of* p.

$\therefore gcd(2n, p-1) = 2 \Rightarrow p \mid 2^2 - 1 \quad \therefore p = 3.$

Let $v_3(n) = x$. *By LTE lemma we must have*:

$v_3(2^n + 1) = v_3(2 + 1) + v_3(n) = 1 + x \geq v_3(n^2) = 2x \Rightarrow x = 1.$

Let $n = 3k$ *with* $gcd(3, k) = 1$ *and if* $k = 1$ *then* $n = 3$, *which is a solution* ($\because\ 9 \mid 2^3 + 1$).

Assume $k > 1$ *and let* $\omega(k) = q$ (*Note*: $q > 3$). *Now we have*:

$$q \mid 8^k + 1 \mid 8^{2k} - 1 \text{ and } q \mid 8^{q-1} - 1 \text{ (By FLT)}.$$

$$\Rightarrow q \mid gcd\left(8^{2k} - 1, 8^{q-1} - 1\right) \Rightarrow q \mid 8^{gcd(2k,q-1)} - 1 \overset{gcd(2k,q-1)=2}{=} 8^2 - 1 = 63$$

$\Rightarrow q \mid 63 \quad \therefore q = 7.$

However $2^n + 1 = 8^k + 1 \equiv 2 \ (mod\ 7)$. *A contradiction*! $\quad (\because q = 7 \mid 2^n + 1)$

Therefore, the only solutions are $n = 1, 3$.