ROMANIAN MATHEMATICAL MAGAZINE

If $x, y, z \neq k\pi\ , k \in Z, \cot^2 x + \cot^2 y + \cot^2 z \leq \frac{3}{7}$ then:

$$\frac{1}{2+\cos 2x} + \frac{1}{2+\cos 2y} + \frac{1}{2+\cos 2z} \geq \frac{12}{5}$$

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Solution by Tapas Das-India

$$Let\ \cot^2 x = a, \cot^2 y = b, \cot^2 z = c\ \ then$$

$$a+b+c = \cot^2 x + \cot^2 y + \cot^2 z \leq \frac{3}{7}\ \ (1)$$

$$2+\cos 2x = 2 + \frac{1-\tan^2 x}{1+\tan^2 x} = 2 + \frac{\cot^2 x + 1}{\cot^2 x - 1} = 2 + \frac{a-1}{a+1} = \frac{3a+1}{a+1}$$

$$\sum \frac{1}{2+\cos 2x} = \sum \frac{a+1}{3a+1}$$

Lemma :

$$\forall\ t > 0\ \frac{t+1}{3t+1} \geq \frac{47-49t}{50}$$

Proof:

$$\frac{t+1}{3t+1} \geq \frac{47-49t}{50}$$

$$50(t+1) \geq (3t+1)(47-49t)$$

$$50t + 50 \geq 92t - 147t^2 + 47$$

$$147t^2 - 42t + 3 \geq 0$$

$$3(7t-1)^2 \geq 0\ true$$

$$\frac{1}{2+\cos 2x} + \frac{1}{2+\cos 2y} + \frac{1}{2+\cos 2z} = \sum \frac{1}{2+\cos 2x} =$$

$$= \sum \frac{a+1}{3a+1} \overset{lemma}{\geq} \sum \frac{47-49a}{50} = \frac{141 - 49(a+b+c)}{50} \overset{(1)}{\geq} \frac{141 - 49 \times \frac{3}{7}}{50} = \frac{120}{50} = \frac{12}{5}$$

Equality holds for $a = b = c = \frac{1}{7}$