# *Extrapropositional and intrapropositional logical operators in classical logics*

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***ABSTRACT*** ***In this work, we will consider the extension of logical operators applied on any type of propositional formulas .***
***Due to the generalization of the logical operators on complex propositional formulas, we will surprise it in two conceptually different forms and we'll choose the variant which conserves the classical square of the oppositions .***

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## § 1. *Introduction*

Due to very large development of *Logics* in our days, it was extremely necessary to underline and study several operators , specific to different branches of *Logics* .

Let's notice the fact that in other domains of *Mathematics* , this situation appeared , sooner or later , depending on the maturity of each field . Thus, special chapters regarding operators in *Mathematical Analysis* are well-known (in this discipline appeared even a distinguished section-very developed actually, of *Operators theory*), in *Topology*, in *Algebra*, in *algebraic Geometry* (*homological Algebra*), in *Mathematical Linguistics*, etc.

In spite of the fact that *Logic* is one of the former sciences of the human thinking, only occasionally and partially – and not as a self-study – were treated some intrinsic logical operators (see for example : [2] , [3] , [6] , [7] , [19] ) .

Starting from the importance of the operators in the exposition and the research of field , in their double quality of ways and instruments (see the situation of the disciplines above mentioned) , we investigated logical operators in different logic ways , this essay being one of them . In the previous works [10] , [11] , [14] , [16] we considered the case of the *binary propositional operators* . Those that are defined over formulas $p \omega q$ , where $p , q$ are *atomic propositions* , and $\omega$ is one of the 10 proper *propositional connectives* : $\vee$ – *disjunction* , & - *conjunction* , $\rightarrow$ - *implication* , $\leftarrow$ - *replication* (the converse of implication) , —‹ - *exclusion* (the implication's negation , difference [14], [16] , [17] ) , ›— - *co-exclusion* (the converse of *exclusion* ,

the *replication*'s negation ) , $\longleftrightarrow$ - *equivalence* (the *bi-implication*) ,
$\succ\!\!-\!\!\prec$ - *bi-exclusion* (the *equivalence's negation* , the *independence* [14],[17] ) ,
I - *incompatibility* (the negation of *conjunction*) , **—** - *rejection* (the *disjunction's negations*) . For the connectors $-\!\!\prec$ , $\succ\!\!-\!\!\prec$ , see [10] - [16] .

We reconsider after [14] and [16] , the definition of the *binary propositional operators* .

Let's take $\mathbb{L} = \langle \mathbf{T} , \mathcal{L} , \mathcal{A} , \mathcal{F} , \mathcal{S} , \mathcal{P} \rangle$ , a *propositional logic* , where **T** is a *theory* , $\mathcal{L}$ - the *reference language* in which is included the theory **T** , $\mathcal{A}$ - the set of *atomic propositions* , $\mathcal{F}$ – the set of ***w.f.f.*** , $\mathcal{S}$ – the set of *symbols* : *negation* ( $\neg$ ) and the *binary connectives* , $\mathcal{P} = \mathcal{A} \cup \mathcal{S}$ - the *symbolic propositional language* , where $\mathcal{A} \cap \mathcal{S} = \varnothing$ . We will also note :

$$\mathcal{F}_2 =_{\text{df}} \{ \, p \ \omega \ q \mid p , q \in \mathcal{A} , \ \omega \text{ - binary proper connective} \} .$$

**1.1. *Definition***

If $p , q \in \mathcal{A}$ and $\omega \in \mathcal{S}$ is a *binary connective* , then we call ***(logical) binary operator*** , a function $\mathfrak{o} : \mathcal{F}_2 \longrightarrow \mathcal{F}_2$ , named , defined and noted as it follows :

**(o)** the operator of ***identity*** : $i\,(\mathbf{p}\ \omega\ \mathbf{q}) =_{\text{df}} \mathbf{p}\ \omega\ \mathbf{q}$ ;

**(oo)** the operator of ***negation*** : $n\,(\mathbf{p}\ \omega\ \mathbf{q}) =_{\text{df}} \neg(\mathbf{p}\ \omega\ \mathbf{q}) =: \mathbf{p}\ \overline{\omega}\ \mathbf{q}$ ;

**(ooo)** the operator of ***conjugation*** : $c\,(\mathbf{p}\ \omega\ \mathbf{q}) =_{\text{df}} \neg\mathbf{p}\ \omega \neg\ \mathbf{q} =: \overline{\mathrm{p}}\ \omega\ \overline{\mathrm{q}}$ ;

**(ov)** the operator of ***preconjugation*** : $c^1(\mathbf{p}\ \omega\ \mathbf{q}) =_{\text{df}} \overline{\mathrm{p}}\ \omega\, \mathbf{q}$ ;

**(v)** the operator of ***postconjugation*** : $c^2(\mathbf{p}\ \omega\ \mathbf{q}) =_{\text{df}} \mathbf{p}\, \omega\ \overline{\mathrm{q}}$ ;

**(vo)** the operator of ***duality*** : $d\,(\mathbf{p}\ \omega\ \mathbf{q}) =_{\text{df}} \overline{\mathrm{p}}\ \overline{\omega}\ \overline{\mathrm{q}}$ ;

**(voo)** the operator of ***preduality*** : $d^1(\mathbf{p}\ \omega\ \mathbf{q}) =_{\text{df}} \overline{\mathrm{p}}\,\overline{\omega}\ \mathbf{q}$ ;

**(vooo)** the operator of ***postduality*** : $d^2(\mathbf{p}\ \omega\ \mathbf{q}) =_{\text{df}} \mathbf{p}\ \overline{\omega}\ \overline{\mathrm{q}}$ .

The properties and structures generated by these operators were described in detail in the above quoted works . We only remind the fact that these operators form a commutative group (with the operation of functional composition) , each of the 7 subgroups of order 4 being a *Klein group* ,-and that these operators form a *logical cube* (that above *Klein subgroups* form *logical squares* which appear on the faces of the cube and on their diagonal plans , mutual isotopic ) . In [12] , [13] , [15] , this analysis was extended for different types of *modal formulas* , as well .

In this work, we will consider the extension and the study of *logical operators* applied on any type of *propositional formulas* .

Due to the generalization of the *logical operators* on *complex propositional formulas*, we will surprise it in two conceptually different forms , we'll choose the variant which conserves the *classical square of the oppositions* (the study also under the aspect of the *interpropositional relations* of the formulas affected by the operators will be presented in a future work) .

The importance of this enterprise –corroborated to the heuristical importance of the study itself of the operators (which clarifies a confuse situation in the logical bibliography, even for the most studied operator , *duality* ) , especially consist in developing and supplying logicians with some important manners of work and investigation for the future .

## § 2. *Extrapropositional and intrapropositional logical operators applied on complex formulas*

Now, we will try an extension of the logical operators presented in the former paragraph, studying their behaviour on propositional formulas of arbitrary length.

The duality was the most studied logical operator, see for example: [1] , [2] , [4] , [5] , [7] - [9] , [16] , [18] - [21] .

The majority of the works studying the duality consider this operator only for formulas containing propositional symbols: $\neg$ , & , $\vee$ ; yet the results obtained are not relevant , as will see in the next paragraph .

Now we will make a complete characterization for all the operators previously defined (definition 1.1.) , the aim wanted (and partially realized in this section) being to show that considering several complex formulas , the logical cube associated to them is preserved . In order to get there , we'll firstly make some considerations on the notations and terminology that will be used later .

If we have a *propositional formula* $\boldsymbol{F}$ , which contains the *propositional variables* $p_1 , p_2 , \dots , p_m$ and the *interpropositional connectives* $\omega_1 , \omega_2 , \dots , \omega_n$ , we will mark this by $F ( p_1 , p_2 , \dots , p_m , \omega_1 , \omega_2 , \dots , \omega_n )$ , or more simply $F ( p_i , \omega_j )$ , where , $i \in \mathbf{I} = \{ 1 , 2 , \dots , m \}$ , $j \in \mathbf{J} = \{ 1 , 2 , \dots , n \}$ .

Often, to make evident the main *connective* $\omega$ of the formula , we will note $F ( p_i , \omega_j , \omega )$ . Notice that if a formula has several main *connectives* , then – resulting from the way itself the *propositional formula* are formed , there should be one (and only one - repeated in several positions ) from the *copulative connectives* (jonctions) ; as all these last *connectives* are associative , one of them can be chosen as the main *connective* .

If $\boldsymbol{a} \in \{ i , n , c , d \}$ , we will mark $\boldsymbol{a} (\omega) = \omega^{\boldsymbol{a}}$ , but will make the difference between $\boldsymbol{a}(F)$ and $F^{\boldsymbol{a}}$ , as it follows .

We consider, by convention, the *dual* of an elementary proposition , as the given proposition itself : $\mathbf{p} \in \mathcal{A} \longrightarrow d(\mathbf{p}) = \mathbf{p}^{d} = \mathbf{p}$ and $d(\neg\, \mathbf{p}) = (\neg\, \mathbf{p})^{d} = \neg\, \mathbf{p}$ .

For the *binary formulas*, we know the *dual* of every kind of propositions like this, and they are given in [14].

We begin by underlining the dual of a complex formula and the correlation between *duality* and the frontal negation of the formula , and between duality and negations over variables-argument . Thus, starting from the definition of the *duality* in the case of formulas with two propositional variables ( Def. 1.1. (vo) ),
$\boldsymbol{d}(\mathbf{p}\ \omega\ \mathbf{q}) = \bar{\mathrm{p}}\ \bar{\omega}\ \bar{\mathrm{q}}$ , we can inherently consider (at least) the next two ways of generalization of duality on formulas (with main connective clearly expressed) .

**2.1. *Definition***

If $\boldsymbol{F}(\mathbf{p}_i, \omega_j)$ , $\boldsymbol{G}(\mathbf{q}_h, \mathrm{w}_k)$ are two *propositional formulas* with $\boldsymbol{i}, \boldsymbol{h} \geq \mathbf{2}$ , $\boldsymbol{j}, \boldsymbol{k} \geq \mathbf{1}$ , we call ***propositional duality*** (or *duality of connective*) , the function , $\boldsymbol{d} : \mathscr{F} \longrightarrow \mathscr{F}$ , defined as ,

$$\boldsymbol{d}((\boldsymbol{F}(\mathbf{p}_i, \omega_j)\ \omega\ \boldsymbol{G}(\mathbf{q}_h, \mathbf{w}_k)) =_{df} \neg \boldsymbol{F}(\mathbf{p}_i, \omega_j)\ \bar{\omega}\ \neg \boldsymbol{G}(\mathbf{q}_h, \mathbf{w}_k) .$$

Also, we will call this, the ***extrapropositional duality***, abbreviated ***e-duality.***

We call ***intrapropositional duality*** , (or duality of formula) in short , ***i-duality*** , the function , $^{\boldsymbol{d}} : \mathscr{F} \longrightarrow \mathscr{F}$ , defined as ,

$$((\boldsymbol{F}(\mathbf{p}_i, \omega_j)\ \omega\ \boldsymbol{G}(\mathbf{q}_h, \mathbf{w}_k))^{\boldsymbol{d}} =_{df} \boldsymbol{F}(\bar{\mathrm{p}}_i, \omega_j)\ \bar{\omega}\ \boldsymbol{G}(\bar{\mathrm{q}}_h, \mathbf{w}_k) ,$$

where $\omega_j, \mathbf{w}_k, \omega$ are proper propositional connectives .

Analogously, we define two kinds of conjugation :

**2.2. *Definition***

- ***Extrapropositional conjugation*** **(** ***e-conjugation*** **)** is the function :

$$\boldsymbol{c} : \mathscr{F} \longrightarrow \mathscr{F} , \text{ defined as ,}$$

$$\boldsymbol{c}((\boldsymbol{F}(\mathbf{p}_i, \omega_j)\ \omega\ \boldsymbol{G}(\mathbf{q}_h, \mathbf{w}_k)) =_{df} \neg \boldsymbol{F}(\mathbf{p}_i, \omega_j)\ \omega\ \neg \boldsymbol{G}(\mathbf{q}_h, \mathbf{w}_k) ;$$

- ***intratrapropositional conjugation*** **(** ***i-conjugation*** **)** is the function :

$$^{\boldsymbol{c}} : \mathscr{F} \longrightarrow \mathscr{F} , \text{ defined as ,}$$

$$((\boldsymbol{F}(\mathbf{p}_i, \omega_j)\ \omega\ \boldsymbol{G}(\mathbf{q}_h, \mathbf{w}_k))^{\boldsymbol{c}} =_{df} \boldsymbol{F}(\bar{\mathrm{p}}_i, \omega_j)\ \omega\ \boldsymbol{G}(\bar{\mathrm{q}}_h, \mathbf{w}_k) .$$

It is evident the fact that the operators of *identity* and *negation* , extended to complex formulas are the same for both variants and consist in the invariation of the formulas , respectively in the negation of the main connective of them .

If $\boldsymbol{\Phi}(\mathbf{p}_i, \omega_j)$ , $\boldsymbol{\Phi}(\mathbf{p}_i, \omega_j, \omega) \in \mathscr{F}$ , then directly from the definition and equivalently to its affirmations , we have :

$$\vdash \boldsymbol{\Phi}^{d}(\mathbf{p}_i, \omega_j, \omega) \equiv \boldsymbol{\Phi}(\bar{\mathrm{p}}_i, \omega_j, \bar{\omega}) , \quad (1)$$

$$\vdash \boldsymbol{\Phi}^{c}(\mathbf{p}_i, \omega_j) \equiv \boldsymbol{\Phi}(\bar{\mathrm{p}}_i, \omega_j) , \quad (2)$$

$$\vdash \boldsymbol{\Phi}^{n}(\mathbf{p}_i, \omega_j, \omega) \equiv \boldsymbol{\Phi}(\mathbf{p}_i, \omega_j, \bar{\omega}) . \quad (3)$$

It is also clear that for the elementary and the binary formulas, the two types of dualisation and conjugation coincide.

***2.3. Theorem*** ( *of* ***e - duality*** )

If $\boldsymbol{F}(\mathbf{p_i}, \omega_{\mathbf{j}})$, $i \in \mathrm{I}$ , $j \in \mathrm{J}$, is a *propositional formula* , then :

$$\vdash \ \boldsymbol{d}(\boldsymbol{F}(\mathbf{p_i},\ \omega_{\mathbf{j}})) \equiv \neg \boldsymbol{F}(\bar{\mathrm{p}}_{\mathbf{i}},\ \omega_{\mathbf{j}}^{\mathbf{d}}) \quad . \qquad (4)$$

***Proof*** The proof goes by induction on the number of the appearance of the connectives $\omega_j$ , $j \in \mathrm{J}$ . We will call the number of connectives - ***order*** (*grade*) ***of complexity*** (like in the already used terminology in the theory of algorithms and calculability , Informatics , etc.) . If $\boldsymbol{F}$ is of zero order , then $\boldsymbol{F} = \mathrm{p}$ or $\boldsymbol{F} = \neg\ \mathrm{p}$ :

- if $\boldsymbol{F} = \mathbf{p}$ , then $\boldsymbol{d}(\boldsymbol{F}) = \mathbf{p} \equiv \neg(\neg\ \mathbf{p})$;
- if $\boldsymbol{F} = \neg\mathbf{p}$ , analogously .

We suppose the affirmation to be true for formulas of *order of a complexity* of most $\boldsymbol{n}$ and proof it for formulas of order $\boldsymbol{n} + \mathbf{1}$ . We have two possibilities :

$\underline{\mathbf{A}}$ : $\boldsymbol{F}(\mathbf{p_i}, \omega_{\mathbf{j}}) = \neg \boldsymbol{G}(\mathbf{q_h}, \mathbf{w_k})$ , $\boldsymbol{i, h} \in \mathbf{I}$ , $\boldsymbol{j, k} \in \mathbf{J}$ , $\omega_{\mathbf{j}}$ , $\mathbf{w_k} \in \mathbf{C.\,p.}$ ;

$\underline{\mathbf{B}}$ : $\boldsymbol{F}(\mathbf{p_i}, \omega_{\mathrm{j}}) = \boldsymbol{F}(\mathbf{p_{i'}}, \omega_{j'})\ \omega\ \boldsymbol{F}(\mathbf{p_{i''}},\ \omega_{j''})$ , $\boldsymbol{i'} \in \mathbf{I'}$ , $\boldsymbol{i''} \in \mathbf{I''}$ , $\mathbf{I'} \cup \mathbf{I''} \subseteq \mathbf{I}$ , $\boldsymbol{j'} \in \mathbf{J'}$ , $\boldsymbol{j''} \in \mathbf{J''}$ , $\mathbf{J'} \cup \mathbf{J''} \subseteq \mathbf{J}$ .

For the $\underline{\mathbf{A}}$ variant , we have successively : $\boldsymbol{d}(\boldsymbol{F}(\mathrm{p_i}, \omega_{\mathrm{j}})) = \boldsymbol{d}(\neg \boldsymbol{G}(\mathrm{q_h}, \mathbf{w_k})) \equiv$
$\equiv \neg \boldsymbol{d}(\boldsymbol{G}(\mathbf{q_h},\ \mathbf{w_k})) \equiv \neg\ (\neg \boldsymbol{G}(\bar{q}_{\mathbf{h}},\ \mathbf{w}_{\mathbf{k}}^{\mathrm{d}})) \equiv G(\bar{q}_{\mathrm{h}},\ \mathbf{w}_{\mathbf{k}}^{\mathrm{d}}) \equiv \neg F(\bar{\mathrm{p}}_{\mathrm{i}},\ \omega_{\mathrm{j}}^{\mathrm{d}})$ , which prove the hypothesis on induction .

For the $\underline{\mathbf{B}}$ variant , if $\omega = \vee$ or $\omega = \boldsymbol{\&}$ , with the application of a frontal *negation* and the substitution $\mathbf{p_i} / \bar{\mathrm{p}}_{\mathbf{i}}$ , the demonstration is in Kleene [9] , Novikov [19] (and in most of the treaties of logics ) .

We will analyze now the situation that all the authors omit or expressly avoid (the above quoted authors put even the condition that implication should miss ) . We will prove that this property hold for implication, proof which will be used as a paradigm for the solving in a general case the affirmation (for each proper connective) .

Thus , if $\boldsymbol{F}(\mathrm{p_i},\ \omega_{\mathrm{j}}) = \boldsymbol{F}(\mathrm{p_{i'}}, \omega_{\mathrm{j'}}) \to \boldsymbol{F}(\mathrm{p_{i''}}, \omega_{\mathrm{j''}})$ , as we have

$$\boldsymbol{d}(\boldsymbol{F_1}(\mathrm{p_{i'}}, \omega_{\mathrm{j'}})) \equiv \neg \boldsymbol{F_1}(\bar{\mathrm{p}}_{\mathrm{i'}}, \omega_{\mathrm{j'}}^{\mathrm{d}}) \ , \qquad (\boldsymbol{a})$$

$$\boldsymbol{d}(\boldsymbol{F_2}(\mathrm{p_{i''}}, \omega_{\mathrm{j''}})) \equiv \neg \boldsymbol{F_2}(\bar{\mathrm{p}}_{\mathrm{i''}}, \omega_{\mathrm{j''}}^{\mathrm{d}}) \ , \qquad (\boldsymbol{b})$$

by negation and $\mathbf{p_i} / \bar{\mathrm{p}}_{\mathbf{i}}$ , $\omega_{\mathrm{j}} / \omega_{\mathrm{j}}^{\mathrm{d}}$ , $\boldsymbol{d} \circ \boldsymbol{d} = \boldsymbol{i}$ , we also have :

$$\boldsymbol{F_1}(\bar{\mathrm{p}}_{\mathrm{i'}}, \boldsymbol{\omega}_{\mathbf{j'}}^{\mathbf{d}}) \equiv \neg \boldsymbol{d}(\boldsymbol{F_1}(\mathrm{p_{i'}}, \omega_{\mathrm{j'}})) \ , \qquad (\boldsymbol{a'})$$

$$\boldsymbol{F_2}(\bar{\mathrm{p}}_{\mathrm{i''}}, \boldsymbol{\omega}_{\mathbf{j''}}^{\mathbf{d}}) \equiv \neg \boldsymbol{d}(\boldsymbol{F_2}(\mathrm{p_{i''}}, \omega_{\mathrm{j''}})) \ . \qquad (\boldsymbol{b'})$$

We obtain the next chain of equivalences:

$$F(\bar{p}_i, \omega_j^d) \equiv \neg(F_1(\bar{p}_{i'}, \omega_{j'}^d) \to F_2(\bar{p}_{i''}, \omega_{j''}^d)) \equiv$$

$$\equiv \neg(\neg d(F_1(p_{i'}, \omega_{j'})) \to \neg d(F_2(p_{i''}, \omega_{j''}))) \equiv$$

$$\equiv \neg(d(F_1(p_{i'}, \omega_{j'})) \leftarrow d(F_2(p_{i''}, \omega_{j''}))) \equiv$$

$$\equiv d(F_1(p_{i'}, \omega_{j'})) \succ\!\!- d(F_2(p_{i''}, \omega_{j''})) \equiv$$

$$\equiv d(F_1(p_{i'}, \omega_{j'}) \to d(F_2(p_{i''}, \omega_{j''})) \equiv d(F(p_i, \omega_j)) \;.$$

For the general case, there also holds the relations (a') , (b') – in the case of the assumption of induction , thus for the case of the *order of complexity* equal to $\boldsymbol{n+1}$ , we have the equivalences :

$$F(\bar{p}_i, \omega_j^d) \equiv \neg(F_1(\bar{p}_{i'}, \omega_{j'}^d)\ \omega\ F_2(\bar{p}_{i''}, \omega_{j''}^d)) \equiv$$

$$\equiv (\ d(F_1(p_{i'}, \omega_{j'}))\ \omega\ d(F_2(p_{i''}, \omega_{j''}))) \equiv$$

$$\equiv d(F_1(p_{i'}, \omega_{j'}))\ \omega^d\ d(F_2(p_{i''}, \omega_{j''})) \equiv d(F(p_i, \omega_j)) \;,$$

which ends the proof completely .

**2.4. *Corollary*** ( the property of ***e-conjugation*** )

If $F(p_I, \omega_j)$ is a *formula of propositional calculus* , then hold :

$$\vdash c(F(p_i, \omega_j)) \equiv F(\bar{p}_i, \omega_j^d),\ (i \in I, j \in J)\,. \qquad \textbf{(4)}$$

The ***Proof*** follows from the application of the negation in the equivalence established in the preceding theorem and from the compositions $\boldsymbol{n \circ c = d}$ , $\boldsymbol{n \circ n = i}$ :

$$c(F(p_i, \omega_j)) \equiv (n \circ d)(F(p_i, \omega_j)) \equiv$$

$$\equiv \neg(\neg F(\bar{p}_i, \omega_j^d)) \equiv F(\bar{p}_i, \omega_j^d) \;.$$

**2.5. *Corollary***

**a)** If $F(p_i, \omega_j)$ is a *tautology* in PC , then $d(F(p_i, \omega_j))$ is a *contradiction* and if $F(p_i, \omega_j)$ is a *contradiction* in PC , then $c(F(p_i, \omega_j))$ is a *tautology* **.**

**b)** If $F(p_i, \omega_j)$ is a *tautology* ( *contradiction* ) , then $c(F(p_i, \omega_j))$ is a *tautology* **(** *contradiction* **) .**

**2.6. *Corollary***

If $\vdash F(p_i, \omega_j) \equiv G(q_h, w_k)$ , $i, h \in I$ , $j, k \in J$ and $a \in \{ i, n, c, d \}$ , then $a(F(p_i, \omega_j)) \equiv a(G(q_h, w_k))$ .

**2.7. Theorem** ( *of i - duality* )

If $F_1(p_{i'}, \omega_{j'})$ , $F_2(p_{i''}, \omega_{j''})$ are two propositional formulas , then :

$$\vdash \ (F_1(p_{i'}, \omega_{j'})\ \omega\ F_2(p_{i''}, \omega_{j''}))^d \equiv F_1^c(p_{i'}, \omega_{j'})\ \omega^n\ F_2^c(p_{i''}, \omega_{j''}) \ , \qquad (6)$$

$$\equiv F_1^d(p_{i'}, \omega_{j'})\ \omega^d\ F_2^d(p_{i''}, \omega_{j''}) \ , \qquad (7)$$

$i' \in I'$ , $i'' \in I''$ , $I' \cup I'' \subseteq I$ , $j' \in J'$ , $j'' \in J''$ , $J' \cup J'' \subseteq J$ .

**Proof**

Using the definition of *i-duality* , we have the following equivalences :

$$[F_1(p_{i'}, \omega_{j'})\ \omega\ F_2(p_{i''}, \omega_{j''})]^d \equiv F_1(p_{i'}, \omega_{j'})\ \overline{\omega}\ F_2(p_{i''}, \omega_{j''}) \overset{(2)}{\equiv}$$

$$\overset{(2)}{\equiv} F_1^c(p_{i'}, \omega_{j'})\ \overline{\omega}\ F_2^c(p_{i''}, \omega_{j''}) \equiv F_1^{nod}(p_{i'}, \omega_{j'})\ \overline{\omega}\ F_2^{nod}(p_{i''}, \omega_{j''}) \equiv$$

$$\equiv \neg F_1^d(p_{i'}, \omega_{j'})\ \overline{\omega}\ \neg F_2^d(p_{i''}, \omega_{j''}) \equiv F_1^d(p_{i'}, \omega_{j'})\ \omega^d F_2^d(p_{i''}, \omega_{j''}) \ .$$

At the last equivalence we applied the definition of the *connective*

$$\omega^d = d(\omega) = \neg\ \overline{\omega}\ \neg \qquad .$$

In practical applications, the next results are even more useful .

**2.8. Corollary**

For a formula $F(p_i, \omega_j)$ from PC , it hold the equivalence

$$\vdash \ F^d(p_i, \omega_j) \equiv F(p_i, \omega_j^d) \ , \quad i \in I \ , \ j \in J \ . \qquad (8)$$

**Proof**

The proof follows by successive application of the theorem of i-duality on the component parts - until the *atomary propositions* $p_i$ , of the *formula* F , for which we have $p_i^d = p_i$ .

**2.9. Theorem** ( *of i - conjugation* )

For two formulas $F_1(p_{i'}, \omega_{j'})$ , $F_2(p_{i''}, \omega_{j''})$ from PC ,

$$\vdash \ (F_1(p_{i'}, \omega_{j'})\ \omega\ F_2(p_{i''}, \omega_{j''}))^c \equiv F_1^c(p_{i'}, \omega_{j'})\ \omega\ F_2^c(p_{i''}, \omega_{j''}) \ , \qquad (9)$$

$$\equiv F_1^d(p_{i'}, \omega_{j'})\ \omega^c\ F_2^d(p_{i''}, \omega_{j''}) \ , \qquad (10)$$

$i' \in I'$ , $i'' \in I''$ , $I' \cup I'' \subseteq I$ , $j' \in J'$ , $j'' \in J''$ , $J' \cup J'' \subseteq J$ .

**Proof**

The proof can be done either directly or deriving it from the *theorem of i-duality*, because of the operatorial relation $c = n \circ d$ , as it follows :

$$\vdash \ (F_1(p_{i'}, \omega_{j'})\ \omega\ F_2(p_{i''}, \omega_{j''}))^c \equiv (F_1(p_{i'}, \omega_{j'})\ \omega\ F_2(p_{i''}, \omega_{j''}))^{nod} \equiv$$

$\equiv \boldsymbol{n}\,(F_1\,(\mathbf{p}_{i'}\,,\,\omega_{j'})\,\omega\,F_2\,(\mathbf{p}_{i''},\,\omega_{j''}))^{d} \equiv$ (using Theorem 2.5.) $\equiv$

$$\equiv \begin{cases} n\,(F_1^{c}\,(\mathbf{p}_{i'},\,\omega_{j'})\,\omega^{n} F_2^{c}\,(\mathbf{p}_{i''},\,\omega_{j''})\,) \equiv F_1^{c}\,(\mathbf{p}_{i'},\omega_{j'})\,\omega^{n} F_2^{c}\,(\mathbf{p}_{i''},\omega_{j''}) \\ \\ n\,(F_1^{d}\,(\mathbf{p}_{i'},\,\omega_{j'})\,\omega^{d} F_2^{d}(\mathbf{p}_{i''},\,\omega_{j''})\,) \equiv F_1^{d}\,(\mathbf{p}_{i'},\,\omega_{j'})\,\omega^{c} F_2^{d}\,(\mathbf{p}_{i''},\,\omega_{j''}) \end{cases} .$$

Besides the formula of recurrence in the theorem of i - conjugation , we can also give global expressions of the behaviour of the operator of conjugation , as in Corollary 2.6. , for *duality*.

**2.10. *Corollary***

For formulas $F\,(\mathbf{p}_i\,,\,\omega_j)\,,\,F\,(\mathbf{p}_i\,,\,\omega_j\,,\,\omega)\ ,\,(i \in \mathbf{I}\ ,\ j \in \mathbf{J}\ )$ of the propositional calculus , we have :

$$(a)\quad \vdash\ F^{c}\,(\mathbf{p}_i\,,\,\omega_j) \equiv F\,(\bar{\mathrm{p}}_i\,,\,\omega_j)\quad ; \qquad \mathbf{(11)}$$

$$(b)\quad \vdash\ F^{c}\,(\mathbf{p}_i\,,\,\omega_j\,,\,\omega) \equiv F\,(\mathbf{p}_i,\ \omega_j^{d},\ \omega^{c})\quad . \qquad \mathbf{(12)}$$

***Proof***

(*a*) The affirmation results from the first equivalence of the theorem of i-conjugation , by successive use on the component parts up to atoms and here , for $\mathrm{p}_i \in \mathcal{A}$ , we have $\bar{\mathrm{p}}_i^{\,c} = \bar{\mathrm{p}}_i$ , $(i \in \mathbf{I})$ .

(*b*) The affirmation results from the second affirmation equivalence in the theorem of *i-conjugation* and Corollary 2.6.

Returning to dualisation , we can also enunciate , with the up-mentioned corollary and the equivalence (6) :

**2.11. *Corollary***

If $F\,(\mathbf{p}_i\,,\ \omega_j\ ,\ \omega)$ is a formula in PC , then we have :

$$\vdash\ F^{d}\,(\mathbf{p}_i\,,\ \omega_j\ ,\omega) \equiv F\,(\bar{\mathrm{p}}_i,\ \omega_j,\ \omega^{n})\quad ,\ i \in \mathbf{I}\ \ ,\ j \in \mathbf{J}\ . \qquad \mathbf{(13)}$$

From this corollary follows another one , regarding the *i-duality* :

**2.12. *Corollary***

If $F\,(\mathbf{p}_i\,,\omega_j)$ is a *tautology* (*contradiction*) of the propositional calculus , with $i \in \mathbf{I}$ , $j \in \mathbf{J} \neq \varnothing$ , then $F^{d}\,(\mathbf{p}_i\,,\omega_j)$ is a *contradiction* (*tautology*) .

***Proof***

Indeed , if $F\,(\mathbf{p}_i\,,\ \omega_j\ ,\ \omega)$ is a tautology in PC , then $F\,(\bar{\mathrm{p}}_i,\ \omega_j,\ \omega)$ is a tautology , too ; thus with (13) , $F^{d}\,(\mathbf{p}_i\,,\ \omega_j\ ,\omega) \equiv F\,(\bar{\mathrm{p}}_i,\ \omega_j,\ \omega^{n})$ is a contradiction .

In the same way follows from 2.8($a$) , the affirmation :

**2.13. *Corollary***

If $\boldsymbol{F}(\mathbf{p}_i, \omega_j, \omega)$ is a *formula* in PC , then we have :

$$\vdash \boldsymbol{F}(\mathbf{p}_i, \omega_j) \;\Leftrightarrow\; \vdash \boldsymbol{F}^c(\mathbf{p}_i, \omega_j) \;,\; i \in \mathbf{I} \;,\; j \in \mathbf{J}\,. \qquad \textbf{(14)}$$

We will show now a final application of the operators of *i-duality* .

**2.12. *Corollary***

**(*a*)** If $F_1(\mathbf{p}_{i'}, \omega_{j'})$ , $F_2(\mathbf{p}_{i''}, \omega_{j''})$ are two *formulas* of PC and $\alpha$ a (*training*) *connective* : $\alpha \in \{ \rightarrow, \leftarrow, \multimap, \rightarrowtail \}$ , then ,

$$\vdash (F_1(\mathbf{p}_{i'}, \omega_{j'})\, \alpha\, F_2(\mathbf{p}_{i''}, \omega_{j''})) \;\Leftrightarrow\; \vdash (F_2^d(\mathbf{p}_{i''}, \omega_{j''})\, \alpha\, F_1^d(\mathbf{p}_{i'}, \omega_{j'})) \;; \qquad \textbf{(15)}$$

**(*b*)**

$$(F_1(\mathbf{p}_{i'}, \omega_{j'}) \longleftrightarrow F_2(\mathbf{p}_{i''}, \omega_{j''})) \;\Leftrightarrow\; \vdash (F_2^d(\mathbf{p}_{i''}, \omega_{j''}) \longleftrightarrow F_1^d(\mathbf{p}_{i'}, \omega_{j'})) \;; \qquad \textbf{(16)}$$

**(*c*)**

$$(F_1(\mathbf{p}_{i'}, \omega_{j'}) \rightarrowtail\!\!\!\!\prec F_2(\mathbf{p}_{i''}, \omega_{j''})) \;\Leftrightarrow\; \vdash (F_2^d(\mathbf{p}_{i''}, \omega_{j''}) \rightarrowtail\!\!\!\!\prec F_1^d(\mathbf{p}_{i'}, \omega_{j'}))\,. \qquad \textbf{(17)}$$

**Proof**

(***a***) If $\vdash (F_1(\mathbf{p}_{i'}, \omega')\, \alpha\, F_2(\mathbf{p}_{i''}, \omega_{j''}))$ , then with 2.12. , 2.7. ( *via* the completeness theorem for PC ) , we have :

$$\vdash \neg (F_1(\mathbf{p}_{i'}, \omega_{j'})\, \alpha\, F_2(\mathbf{p}_{i''}, \omega_{j''}))^d \quad \Leftrightarrow$$

$$\Leftrightarrow \quad \vdash \neg\, F_1^d(\mathbf{p}_{i'}, \omega_{j'})\, \alpha^d F_2^d(\mathbf{p}_{i''}, \omega_{j''}) \quad \Leftrightarrow$$

$$\Leftrightarrow \quad \vdash F_1^d(\mathbf{p}_{i'}, \omega_{j'})\, \alpha^{don} F_2^d(\mathbf{p}_{i''}, \omega_{j''}) \quad \Leftrightarrow$$

$$\Leftrightarrow \quad \vdash F_1^d(\mathbf{p}_{i'}, \omega_{j'})\, \alpha^c F_2^d(\mathbf{p}_{i''}, \omega_{j''}) \;,$$

and since $\rightarrow^c = \leftarrow$ , $\leftarrow^c = \rightarrow$ , $\multimap^c = \rightarrowtail$ , $\rightarrowtail^c = \multimap$ , the affirmations result .

($b$) , ($c$) The proofs follow from ($a$) .

## § 3. *Correlations between logical e-operators and i-operators . Option for i-operators .*

In the previous paragraph we established several relations concerning logical *e-operators* and *i-operators* applied on some complex formulas in propositional calculus.

We will now establish some interrelations between the two types of logical operators . For this , we resume and renumber (for the comfortable considerations in this paragraph) the relations of global characterization of the operator’s behaviour on propositional formulas :

$$\vdash \quad \boldsymbol{d}(\boldsymbol{F}(\mathbf{p}_i,\ \omega_j)) \equiv \boldsymbol{F}(\bar{\mathrm{p}}_i,\ \omega_j^{\mathrm{d}}) \quad ; \qquad (1)$$

$$\vdash \quad \boldsymbol{c}(\boldsymbol{F}(\mathbf{p}_i,\ \omega_j)) \equiv \boldsymbol{F}(\bar{\mathrm{p}}_i,\ \omega_j^{\mathrm{d}}) \quad ; \qquad (2)$$

$$\vdash \quad n(F(\mathrm{p}_i,\ \omega_j,\omega)) \equiv F^{n}(\mathrm{p}_i,\omega_j,\omega) \equiv F(\mathrm{p}_i,\omega_j,\omega^{n}) \ ; \qquad (3)$$

$$\vdash \quad \boldsymbol{F}^{\boldsymbol{d}}(\mathbf{p}_i,\ \omega_j) \equiv \boldsymbol{F}(\mathbf{p}_i,\ \omega_j^{\mathrm{d}}) \quad ; \qquad (4)$$

$$\vdash \quad \boldsymbol{F}^{\boldsymbol{d}}(\mathbf{p}_i,\ \omega_j,\omega) \equiv \boldsymbol{F}(\bar{\mathrm{p}}_i,\ \omega_j,\ \omega^{n}) \quad ; \qquad (5)$$

$$\vdash \quad \boldsymbol{F}^{\boldsymbol{c}}(\mathbf{p}_i,\ \omega_j) \equiv \boldsymbol{F}(\bar{\mathrm{p}}_i,\ \omega_j) \quad ; \qquad (6)$$

$$\vdash \quad \boldsymbol{F}^{\boldsymbol{c}}(\mathbf{p}_i,\ \omega_j,\omega) \equiv \boldsymbol{F}(\mathbf{p}_i,\ \omega_j^{\mathrm{d}},\ \omega^{c}) \quad . \qquad (7)$$

$(\boldsymbol{i} \in \mathbf{I}\ ,\ \boldsymbol{j} \in \mathbf{J}\ )$ .

From the relations (4) and (6) , we have :

**3.1. *Theorem***

If $\boldsymbol{F}(\mathbf{p}_i,\ \omega_j)$ is a propositional formula , $\boldsymbol{i} \in \mathbf{I}$ , $\boldsymbol{j} \in \mathbf{J}$ , then ,

$$\vdash \quad \boldsymbol{F}^{\boldsymbol{n}}(\mathrm{p}_i,\omega_j) \equiv \boldsymbol{F}(\bar{\mathrm{p}}_i,\ \omega_j^{\mathrm{d}}) \quad . \qquad (8)$$

***Proof***

Indeed , $\boldsymbol{F}^{\boldsymbol{n}}(\mathbf{p}_i,\omega_j) \equiv \boldsymbol{F}^{\boldsymbol{d} o \boldsymbol{c}}(\mathbf{p}_i,\omega_j) \equiv F^{d}(\bar{\mathrm{p}}_i,\omega_j) \equiv \boldsymbol{F}(\bar{\mathrm{p}}_i,\omega_j^{\mathrm{d}})$ .

The same relation can be obtained if relations (5) and (7) are used . With (5) and (6) or (4) and (7) , relation (3) is recovered .

We can also notice that from (1) and (6) :

$$\vdash \quad \boldsymbol{d}(\boldsymbol{F}(\mathbf{p}_i,\ \omega_j)) \equiv \boldsymbol{F}^{\boldsymbol{c}}(\mathbf{p}_i,\omega_j^{\mathrm{d}}) \equiv \boldsymbol{F}^{\boldsymbol{d}}(\mathbf{p}_i,\omega_j^{\mathrm{d}}) \quad ,$$ therefore ,

$$\vdash \quad \boldsymbol{d}(\boldsymbol{F}(\mathbf{p}_i,\ \omega_j)) \equiv \boldsymbol{F}^{\boldsymbol{d}}(\mathbf{p}_i,\omega_j^{\mathrm{d}}) \quad , \qquad (9)$$

and (2) and (6) , we have :

$$\vdash \quad \boldsymbol{c}(\boldsymbol{F}(\mathrm{p}_i,\ \omega_j)) \equiv \boldsymbol{F}^{\boldsymbol{c}}(\mathrm{p}_i,\omega_j^{\mathrm{d}}) \quad . \qquad (10)$$

From (2) and (8) , result :

$$\vdash \quad n(F(p_i,\ \omega_j)) \equiv F^n(p_i, \omega_j) \equiv c(F(p_i, \omega_j)) \qquad (11)$$

and from relations (9) and (4) , - for example , we have :

$$\vdash \quad d(F(p_i,\ \omega_j)) \equiv F^d(p_i, \omega_j^d) \equiv F(p_i, \omega_j) \ , \qquad (12)$$

following that *e-operators* ***i , n , , c , d*** , do not set up a consistent square (*vide* [2] , [7] , [14] , [16] ) when they is extended to arbitrary propositional statements ; but they form a degenerated square , where ***d = i*** and ***n = c*** .

From (12) results that e-duality applied to a formula , produce the same formula , usually more complicated , if more than half of the component elementary propositions are not negated, because – according to relation (1) , propositional variables change into their negations and connectives into dual connectives .

This is in fact the generalization of *De Morgan's law* (affected by a negation , the direct form being (11) ) . In fact, the only relation with the duality consists in the fact that by negating a formula results dual connectives of those in the formula .

We could see by this that classical duality, studied by most logical treatises, is a non-operative fiction , since it doesn't lead to new logic laws – and thus , all the results regarding e-duality return to banal logical identities .

Also , the fact that we don't obtain anymore a consistent logical square for logical e-operators, makes this group of operators rather uninteresting and compromises the notions of e-duality and e-conjugation . The only notable profit is the generalized *De Morgan's law* ,

$$\vdash \quad n(F(p_i, \omega_j)) \equiv F(\bar{p}_i, \omega_j^d) \quad , \qquad (13)$$

obtained from (11) and (12) .

With the relations (11) and (12) we also have an immediate answer for Corollary 2.3. . (Let's notice that the affirmations in 2.3. , don't intervene in none of the relations which lead to (11) and (12) , meaning that the demonstration presented above is not vicious ! ) .

Already mentioned , we note the non-relevance of e-operators , in case of e-conjugation and of e-duality .

We cannot say the same thing about i-operators which intervene in (3)-(7) relations . These preserve the logical square and effectively lead to new logical relations (starting from certain given relations), increasing through the register and content of the logic . It's easy to notice that the passage from one given formula to another through one i-operator is done quite economical. Especially, the passing is done just by changing (affecting) the interpropositional connectives , grace to the (3) , (4) and (7) relations , without affecting propositional variables , their order in the formula, the positions of parentheses , etc. Since the i-operators , the duality and conjugation ideas , established in the cases of binary formulas , acquired a general and generalized form - and they deserve our attention .

Now , we consider the relations which deal with i-operators actions on $F_1(\, p_{i'}, \omega_{j'}) \;\omega\; F_2(\, p_{i''}, \omega_{j''})$ formulas type , formulas which be notated further on , shortly by $F_1 \;\omega\; F_2$ .

In the previous paragraph we defined or established some such formulas, another will be obtained further on.

We remind these recurrent relations , this time alphanumeric numbered , for a much easier reference to them :

$$\vdash \;(F_1 \;\omega\; F_2)^i \equiv F_1 \omega\; F_2 \quad , \qquad \text{(I 1)}$$

$$\vdash \;(F_1 \;\omega\; F_2)^c \equiv F_1^c \omega\; F_2^c \quad , \qquad \text{(C1)}$$

$$\vdash \;(F_1 \;\omega\; F_2)^c \equiv F_1^d \omega^c F_2^d \quad , \qquad \text{(C2)}$$

$$\vdash \;(F_1 \;\omega\; F_2)^d \equiv F_1^d \omega^d F_2^d \quad , \qquad \text{(D1)}$$

$$\vdash \;(F_1 \;\omega\; F_2)^d \equiv F_1^c \;\omega^n F_2^c \quad , \qquad \text{(D2)}$$

$$\vdash \;(F_1 \;\omega\; F_2)^n \equiv F_1 \;\omega^n F_2 \quad , \qquad \text{(N1)}$$

### 3.2. *Proposition*

If $F_1$ , $F_2$ are two formulas of propositional calculus , then :

$$\vdash \;(F_1 \;\omega\; F_2)^n \equiv F_1^n \omega^d F_2^n \; . \qquad \text{(N2)}$$

***Proof***

We have the following PC- equivalences :

$$\vdash \;(F_1 \;\omega\; F_2)^n \equiv (F_1 \;\omega\; F_2)^{c \circ d} \equiv (F_1^c \omega\, F_2^c)^d \equiv (F_1^c)^d \omega^d (F_2^c)^d \equiv$$

$\equiv F_1^n \omega^d F_2^n$ . They are also possible another proofs .

We notice that for each operator – excepting the identity, we obtained a couple of expressions. It is expected that for identity operator to find out another formula , too .

Indeed , we have the following :

### 3.3. *Proposition*

(*a*) If $F_1$ , $F_2$ are PC's formulas , then we have :

$$\vdash \;F_1 \;\omega\; F_2 \equiv F_1^n \omega^c F_2^n \; ; \qquad \text{(I 2)}$$

(*b*)

$$\vdash \;F^i(\mathrm{p}_i, \omega_j, \omega) \equiv F(\bar{\mathrm{p}}_i, \omega_j^d, \omega^c) \quad . \qquad \text{(14)}$$

***Proof***

(*a*) We have the next proof sequence :

$$\vdash \ F_1 \ \omega \ F_2 \ \equiv \ (F_1 \ \omega \ F_2)^{c\,o\,c} \equiv (F_1^d \, \omega^c \, F_2^d)^c \ \equiv \ F_1^{d\circ c} \, \omega^c \, F_2^{d\circ c} \equiv$$

$$\equiv \ F_1^n \, \omega^c \, F_2^n \quad .$$

$$(b) \quad \vdash \ F^i(p_i, \ \omega_j, \omega) \equiv F_1(p_{i'}, \omega_{j'}) \ \omega \ F_2(p_{i''}, \omega_{j''}) \overset{(a)}{\equiv}$$

$$\equiv \ F_1^n(p_{i'}, \omega_{j'}) \ \omega^c \ F_2^n(p_{i''}, \omega_{j''}) \equiv F_1(\bar{p}_{i'}, \omega^d_{j'}) \ \omega^c \ F_2(\bar{p}_{i''}, \omega^d_{j''}) \equiv$$

$$\equiv \ F(\bar{p}_i, \omega^d_j, \omega^c) \quad .$$

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