

ROMANIAN MATHEMATICAL MAGAZINE

Find all $A \in M_2(\mathbb{R})$ such that $A^2 = -A^T$

Proposed by Nguyen Hung Cuong-Vietnam

Solution by Nawar Alasadi-Iraq

$$A^2 = -A^T \Rightarrow (A^2)^T = (-A^T)^T \Rightarrow (A^T)^2 = -A$$

$$\text{Substitute } A^T = -A^2 \Rightarrow (-A^2)^2 = -A \Rightarrow A^4 + A = 0 \Rightarrow A(A^3 + I) = 0$$

Minimal polynomial $m_A(x)$ divides $x(x+1)(x^2-x+1)$. Since $A \in M_2(\mathbb{R})$, characteristic polynomial $P_A(x)$ is real of degree 2.

Case 1: $P_A(x) = x^2 \Rightarrow m_A(x) = x \Rightarrow A = \begin{pmatrix} 0 & 0 \\ 0 & 0 \end{pmatrix}$

Case 2: $P_A(x) = (x+1)^2 \Rightarrow m_A(x) = x+1 \Rightarrow A = \begin{pmatrix} -1 & 0 \\ 0 & -1 \end{pmatrix}$

Case 3: $P_A(x) = x(x+1) \Rightarrow A^2 + A = 0 \Rightarrow A^2 = -A$. From $A^2 = -A^T \Rightarrow -A = -A^T \Rightarrow A = A^T$. Let $A = \begin{pmatrix} a & b \\ b & d \end{pmatrix}$. $\text{tr}(A) = -1 \Rightarrow d = -1 - a$. $\det(A) = 0 \Rightarrow a(-1-a) - b^2 = 0 \Rightarrow a^2 + a + b^2 = 0$.

$$A = \begin{pmatrix} a & b \\ b & -1-a \end{pmatrix} \quad \text{where } a^2 + a + b^2 = 0$$

Case 4: $P_A(x) = x^2 - x + 1 \Rightarrow A^2 - A + I = 0 \Rightarrow A^2 = A - I$. From $A^2 = -A^T \Rightarrow A - I = -A^T \Rightarrow A + A^T = I$. Let $A = \begin{pmatrix} a & b \\ c & d \end{pmatrix} \Rightarrow \begin{pmatrix} 2a & b+c \\ b+c & 2d \end{pmatrix} = \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix} \Rightarrow$

$$a = d = \frac{1}{2}, c = -b. \det(A) = 1 \Rightarrow \frac{1}{4} + b^2 = 1 \Rightarrow b = \pm \frac{\sqrt{3}}{2}.$$

$$A = \begin{pmatrix} \frac{1}{2} & \pm \frac{\sqrt{3}}{2} \\ \mp \frac{\sqrt{3}}{2} & \frac{1}{2} \end{pmatrix}$$

Solution Set:

$$\left\{ \begin{pmatrix} 0 & 0 \\ 0 & 0 \end{pmatrix}, \begin{pmatrix} -1 & 0 \\ 0 & -1 \end{pmatrix}, \begin{pmatrix} \frac{1}{2} & \pm \frac{\sqrt{3}}{2} \\ \mp \frac{\sqrt{3}}{2} & \frac{1}{2} \end{pmatrix} \right\} \cup \left\{ \begin{pmatrix} a & b \\ b & -1-a \end{pmatrix} \mid a^2 + a + b^2 = 0 \right\}$$