# ROMANIAN MATHEMATICAL MAGAZINE

**If $A, B \in M_2(\mathbb{R}); A^2 - AB = 8I_2; B^2 - BA = I_2$ then find:**

$$M = \det A + \det B.$$

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$$A^2 - AB = 8I_2 \Rightarrow A(A - B) = 8I_2 \Rightarrow \det\big(A(A - B)\big) = \det(8I_2)$$

$$\det A \cdot \det(A - B) = 64 \neq 0 \Rightarrow (\exists)A^{-1}$$

$$A^2 - AB = 8I_2 \Rightarrow A^{-1}(A^2 - AB) = 8A^{-1}I_2$$

$$A^{-1}A^2 - A^{-1}AB = 8A^{-1} \Rightarrow A - B = 8A^{-1} \Rightarrow A = B + 8A^{-1}$$

$$B^2 - BA = I_2 \Rightarrow B^2 - B(B + 8A^{-1}) = I_2$$

$$B^2 - B^2 - 8BA^{-1} = I_2 \Rightarrow -8BA^{-1} = I_2 \Rightarrow -8BA^{-1}A = I_2 \cdot A$$

$$\Rightarrow -8BI_2 = A \Rightarrow A = -8B$$

$$B^2 - BA = I_2 \Rightarrow B^2 - B(-8B) = I_2 \Rightarrow 9B^2 = I_2$$

$$B^2 = \frac{1}{9}I_2 \Rightarrow \det(B^2) = \det\left(\frac{1}{9}I_2\right) \Rightarrow (\det B)^2 = \frac{1}{81} \Rightarrow \det B = \frac{1}{9}$$

$$A = -8B \Rightarrow \det A = \det(-8B) \Rightarrow \det A = (-8)^2 \det B$$

$$\det A = 64 \cdot \frac{1}{9} = \frac{64}{9}$$

$$M = \det A + \det B = \frac{64}{9} + \frac{1}{9} = \frac{65}{9}$$