ROMANIAN MATHEMATICAL MAGAZINE

Let a, b, c be the roots of the polnomial $P(x) = x^3 - yx^2 + x - 5$. Find

$$\sum \frac{1}{a^2 - ya}$$

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Considering that these equalities hold according to Vieta's formulas ...

$$\begin{cases} \sum a = y \\ \sum ab = 1 \\ abc = 5 \end{cases}$$

$$\begin{cases} a^2 - ya = a(a-y) = -a(b+c) \\ b^2 - yb = b(b-y) = -b(a+c) \\ c^2 - yc = c(c-y) = -c(a+b) \end{cases} \Rightarrow \sum \frac{1}{a^2 - ya} = -\left[\frac{1}{a(b+c)} + \frac{1}{b(a+c)} + \frac{1}{c(a+b)}\right]$$

$$= -\left[\frac{1}{ab+ac} + \frac{1}{ab+bc} + \frac{1}{ac+bc}\right] = -\left[\frac{1}{1-bc} + \frac{1}{1-ac} + \frac{1}{1-ab}\right]$$

$$= -\frac{(1-ac)(1-ab) + (1-bc)(1-ab) + (1-bc)(1-ac)}{(1-bc)(1-ac)(1-ab)}$$

$$= -\frac{3 - 2\sum ab + abc\sum a}{1 - \sum ab + abc\sum a - (abc)^2} = -\frac{3 - 2 + 5y}{1 - 1 + 5y - 25} = \frac{1 + 5y}{25 - 5y}$$