ROMANIAN MATHEMATICAL MAGAZINE

NAGEL'S CEVIANS REVISITED (II)

By Bogdan Fuștei-Romania

Let ABC be a triangle with sides a, b, c, semiperimeter

$$s = \frac{a+b+c}{2},$$

and area Δ. Define: $x = \sqrt{s^2 - 2r_a h_a}, \qquad y = \sqrt{s^2 - 2r_b h_b}.$

We have

$$r_a = \frac{\Delta}{s-a}, \qquad h_a = \frac{2\Delta}{a},$$

hence

$$2r_a h_a = \frac{4\Delta^2}{a(s-a)}.$$

By Heron's formula $\Delta^2 = s(s-a)(s-b)(s-c)$, we obtain

$$x^2 = s^2 - \frac{4s(s-b)(s-c)}{a}.$$

Since

$$(s-b)(s-c) = \frac{a^2 - (b-c)^2}{4},$$

it follows that

$$x^2 = s(s-a) + \frac{s(b-c)^2}{a}.$$

Similarly,

$$y^2 = s(s-b) + \frac{s(a-c)^2}{b}.$$

Set

$$p = s - a, \qquad q = s - b, \qquad t = s - c.$$

Then

$$a = q + t, \quad b = p + t, \quad c = p + q, \quad s = p + q + t,$$
$$b - c = t - q, \qquad a - c = t - p.$$

Therefore

$$x^2 = s\left(p + \frac{(t-q)^2}{q+t}\right), \qquad y^2 = s\left(q + \frac{(t-p)^2}{p+t}\right).$$

Hence

$$x \geq \sqrt{sp}, \qquad y \geq \sqrt{sq}, \qquad xy \geq s\sqrt{pq}.$$

Thus

$$(x+y)^2 \geq s\left(p + q + \frac{(t-q)^2}{t+q} + \frac{(t-p)^2}{t+p} + 2\sqrt{pq}\right).$$

Lemma. For $\mathbf{u}, \mathbf{v} > \mathbf{0}$,

$$\frac{(\mathbf{u}-\mathbf{v})^2}{\mathbf{u}+\mathbf{v}} \geq \left(\sqrt{\mathbf{u}} - \sqrt{\mathbf{v}}\right)^2.$$

Applying it with $(u, v) = (t, q)$ and (t, p) gives

$$\frac{(t-q)^2}{t+q} + \frac{(t-p)^2}{t+p} \geq \left(\sqrt{t} - \sqrt{q}\right)^2 + \left(\sqrt{t} - \sqrt{p}\right)^2.$$

Hence

$$p + q + \frac{(t-q)^2}{t+q} + \frac{(t-p)^2}{t+p} + 2\sqrt{pq} \ \geq p + q + t + \left(\sqrt{p} + \sqrt{q} - \sqrt{t}\right)^2 > p + q + t.$$

Therefore

$$(x+y)^2 > s(p+q+t) = s^2.$$

$$\boxed{x + y > s}$$

But $n_a = \sqrt{s^2 - 2r_a h_a}$ (and analogous)[1] →

$$\mathbf{n_a+n_b > s} \text{ (and analogous)(1)}$$

From $n_a=\sqrt{s^2-2r_ah_a}$ (and analogous)→

$$\mathbf{s > n_a} \text{ (and analogous)(2)}$$

n_a, n_b, n_c-Nagel cevians, from (1) and (2)→

$$\mathbf{n_a+n_b > n_c} \text{ (and analogous) (3)}$$

$$\mathbf{n_a, n_b, n_c} \text{-can be sides of a triangle (4)}$$

From $n_a=\sqrt{s^2-2r_ah_a}$ (and analogous)→$2r_ah_a=s^2-n_a^2=(s-n_a)(s+n_a)$

$\frac{2r_a}{s+n_a}=\frac{s}{h_a}-\frac{n_a}{h_a}$ (and analogous); $ah_a=2sr$→ $\frac{2r_a}{s+n_a}=\frac{a}{2r}-\frac{n_a}{h_a}$

$$\frac{\mathbf{a}}{\mathbf{2r}}=\frac{\mathbf{n_a}}{\mathbf{h_a}}+\frac{\mathbf{2r_a}}{\mathbf{s+n_a}} \text{ (and analogous)(5)}$$

From (3) and (5):

$$\frac{\mathbf{a}}{\mathbf{2r}}>\frac{\mathbf{n_a}}{\mathbf{h_a}}+\frac{\mathbf{2r_a}}{\mathbf{n_a+n_b+n_c}} \text{ (and analogous)(6)}$$

$$\frac{\mathbf{a}}{\mathbf{2r}}>\frac{\mathbf{n_a}}{\mathbf{h_a}}+\frac{\mathbf{2r_a}}{\mathbf{2n_a+n_c}} \text{(and analogous)(7)}$$

$$\frac{\mathbf{a}}{\mathbf{2r}}>\frac{\mathbf{n_a}}{\mathbf{h_a}}+\frac{\mathbf{2r_a}}{\mathbf{2n_a+n_b}} \text{(and analogous)(8)}$$

$2r_ah_a=s^2-n_a^2=(s-n_a)(s+n_a)$→$\frac{2r_ah_a}{s-n_a}=s+n_a$

$$\frac{\mathbf{2r_a}}{\mathbf{s-n_a}}=\frac{\mathbf{a}}{\mathbf{2r}}+\frac{\mathbf{n_a}}{\mathbf{h_a}} \text{(and analogous)(9)}$$

$s-n_a<n_b+n_c-n_a$→ $\frac{1}{s-n_a}>\frac{1}{n_b+n_c-n_a}$ → $\frac{2r_a}{s-n_a}>\frac{2r_a}{n_b+n_c-n_a}$

$$\rightarrow\frac{\mathbf{a}}{\mathbf{2r}}+\frac{\mathbf{n_a}}{\mathbf{h_a}}>\frac{\mathbf{2r_a}}{\mathbf{n_b+n_c-n_a}} \text{ (and analogous)(10)}$$

$\frac{n_a}{h_a}=\frac{\sqrt{4r^2+(b-c)^2}}{2r}$ (and analogous)[2] and from (6),(7),(8) :

$$\frac{\mathbf{n_a+n_b+n_c}}{\mathbf{r}}>\frac{\mathbf{4r_a}}{\mathbf{a-\sqrt{4r^2+(b-c)^2}}} \text{ (and analogous) (11)}$$

$$\frac{\mathbf{2n_a+n_c}}{\mathbf{r}}>\frac{\mathbf{4r_a}}{\mathbf{a-\sqrt{4r^2+(b-c)^2}}} \text{(and analogous) (12)}$$

$$\frac{2n_a+n_b}{r} > \frac{4r_a}{a-\sqrt{4r^2+(b-c)^2}}$$ **(and analogous) (13)**

From $\frac{s}{h_a} = \frac{a}{2r} \rightarrow$

$$\frac{s}{h_a} = \frac{a}{2r} = \frac{s-a}{h_a-2r}$$ **(and analogous)(14)**

$\frac{n_a}{h_a} = \frac{\sqrt{4r^2+(b-c)^2}}{2r} \rightarrow$

$$\frac{n_a}{h_a} = \frac{\sqrt{4r^2+(b-c)^2}}{2r} = \frac{n_a-\sqrt{4r^2+(b-c)^2}}{h_a-2r}$$ **(and analogous)(15)**

$$\frac{s}{h_a} = \frac{a}{2r} \rightarrow \frac{s}{h_a} = \frac{a}{2r} = \frac{s+a}{h_a+2r}$$ **(and analogous)(16)**

$$\frac{n_a}{h_a} = \frac{\sqrt{4r^2+(b-c)^2}}{2r} \rightarrow \frac{n_a}{h_a} = \frac{\sqrt{4r^2+(b-c)^2}}{2r} = \frac{n_a+\sqrt{4r^2+(b-c)^2}}{h_a+2r}$$ **(and analogous)(17)**

From (14) and (5) :

$$\frac{2r_a}{s+n_a} = \frac{s+\sqrt{4r^2+(b-c)^2}-a-n_a}{h_a-2r}$$ **(and analogous) (18)**

From (5) and (16):

$$\frac{2r_a}{s+n_a} = \frac{s+a-\sqrt{4r^2+(b-c)^2}-n_a}{h_a+2r}$$ **(and analogous)(19)**

From (18) and (1):

$$\frac{n_b+n_c+\sqrt{4r^2+(b-c)^2}-a-n_a}{h_a-2r} > \frac{2r_a}{s+n_a}$$ **(and analogous)(20)**

From (19) and (1):

$$\frac{n_b+n_c+a-\sqrt{4r^2+(b-c)^2}-n_a}{h_a+2r} > \frac{2r_a}{s+n_a}$$ **(and analogous) (21)**

We use :$\frac{r_a}{r} = \frac{h_a}{h_a-2r} = \frac{n_a}{n_a-\sqrt{4r^2+(b-c)^2}}$ (and analogous) [3]

$$\rightarrow \frac{r_a}{r} = \frac{n_a+h_a}{n_a-\sqrt{4r^2+(b-c)^2}+h_a-2r}$$ **(and analogous)(22)**

Also :$\frac{r_a}{r} = \frac{h_a}{h_a-2r} = \frac{n_a}{n_a-\sqrt{4r^2+(b-c)^2}} = \frac{s}{s-a}$ (and analogous) [3]

$$\rightarrow \frac{r_a}{r} = \frac{n_a+s}{s-a+n_a-\sqrt{4r^2+(b-c)^2}} \text{ (and analogous) (23)}$$

From (1) and (23) :

$$\frac{2n_a+n_b}{s-a+n_a-\sqrt{4r^2+(b-c)^2}} > \frac{r_a}{r} \text{ (and analogous) (24)}$$

$$\frac{n_a+n_b+n_c}{s-a+n_a-\sqrt{4r^2+(b-c)^2}} > \frac{r_a}{r} \text{ (and analogous)(25)}$$

$$\frac{2n_a+n_c}{s-a+n_a-\sqrt{4r^2+(b-c)^2}} > \frac{r_a}{r} \text{ (and analogous) (26)}$$

REFERENCES:

[1]. **Bogdan Fuștei-ABOUT NAGEL'S AND GERGONNE'S CEVIANS (II)-www.ssmrmh.ro**

[2]. **Bogdan Fuştei-ABOUT NAGEL'S AND GERGONNE'S CEVIANS-(VII) -www.ssmrmh.ro**

[3]. **Bogdan Fuștei-NAGEL'S CEVIANS REVISITED-www.ssmrmh.ro**