SSMA-MATH CHALLENGES-VII

DANIEL SITARU - ROMANIA

5570. Suppose $a, b \in \mathbb{C}$ with $|a^2+1| \le 1, |a^4+1| \le 1, |b^3+1| \le 1, |b^6+1| \le 1$. Prove that:

$$|a+b|^2 + |a-b|^2 \le 4$$

Proposed by Daniel Sitaru - Romania

Solution 1 by Michel Bataille, Rouen, France.
We have $|a+b|^2 = (a+b)(\overline{a}+\overline{b}) = |a|^2 + a\overline{b} + \overline{a}b + |b|^2$ and $|a-b|^2 = |a|^2 - a\overline{b} - \overline{a}b + |b|^2$, hence $|a+b|^2 + |a-b|^2 = 2(|a|^2+|b|^2)$. Thus, it suffices to show that $|a|^2 + |b|^2 \le 2$ or even that $|a| \le 1$ and $|b| \le 1$.
If $2|a^2| \le 1$, then certainly $|a| \le 1$. If $2|a|^2 \ge 1$, then using the triangular inequality, we see that

$$1 \ge |a^4+1| = |(a^2+1)^2 - 2a^2| \ge \left| |a^2+1|^2 - 2|a|^2 \right| = 2|a|^2 - |a^2+1|^2$$

so that $2 \ge 1 + |a^2+1|^2 \ge 2|a^2| = 2|a|^2$ and $|a| \le 1$ follows.
Similarly, we have $|b| \le 1$ if $2|b|^3 = |2b^3| \le 1$. If $|2b^3| \ge 1$, then, as above,

$$1 \ge |b^6+1| = |(b^3+1)^2 - 2b^3| \ge \left| |b^3+1|^2 - |2b^3| \right| = |2b^3| - |b^3+1|^2$$

hence $2 \ge 1 + |b^3+1|^2 \ge 2|b|^3$ and $|b| \le 1$ follows.
In any case, we have $|a| \le 1$ and $|b| \le 1$. □

Solution 2 by proposer.

$$2|a^4| = |2a^4| = |(a^2+1)^2 - (a^4+1)| \le$$

$$\le |(a^2+1)^2| + |a^4+1| = |a^2+1|^2 + |a^4+1| \le 1+1 = 2$$

$$\Rightarrow 2|a^4| \le 2 \Rightarrow |a^4| \le 1 \Rightarrow (|a|)^4 \le 1 \Rightarrow |a| \le 1$$

$$2|b^6| = |2b^6| = |(b^3+1)^2 - (b^6+1)| \le$$

$$\le |(b^3+1)^2| + |b^6+1| = |b^3+1|^2 + |b^6+1| \le$$

$$\le 1^2 + 1 = 2 \Rightarrow 2|b^6| \le 2 \Rightarrow |b|^6 \le 1 \Rightarrow |b| \le 1$$

By parallelogram identity:

$$|a+b|^2 + |a-b|^2 = 2(|a|^2+|b|^2) \le 2(1^2+1^2) = 4$$

□

5779. If $0 < a \le b$ then:

$$e^{ab} + e^{\left(\frac{2ab}{a+b}\right)^2} \le e^{\left(\frac{2ab}{a+b}\right)^2} + e^{\left(\sqrt{ab} + \frac{a+b}{2} - \frac{2ab}{a+b}\right)^2}$$

Daniel Sitaru - Romania

Solution 1 by Albert Stadler, Herrliberg, Switzerland.
We divide both sides by $e^{(\frac{2ab}{a+b})^2} > 0$ and get the equivalent inequality
$$e^{\frac{ab(a-b)^2}{(a+b)^2}} + e^{\frac{(a-b)^2(a^2+6ab+b^2)}{4(a+b)^2}} \le 1 + e^{\frac{(a-b)^2(a+4\sqrt{ab}+b)}{4(a+b)}}.$$
Let x be real. The Taylor expansion of the exponential function is
$$e^x = \sum_{k=0}^{\infty} \frac{x^k}{k!.}$$
It is therefore sufficient to prove that
$$\Big(\frac{ab(a-b)^2}{(a+b)^2}\Big)^k + \Big(\frac{(a-b)^2(a^2+6ab+b^2)}{4(a+b)^2}\Big)^k \le \Big(\frac{(a-b)^2(a+4\sqrt{ab}+b)}{4(a+b)}\Big)^k$$
for all $k \ge 1$. This inequality is equivalent to
$$(4ab)^k + (a^2+6ab+b^2)^k \le \Big((a+4\sqrt{ab}+b)(a+b)\Big)^k, \quad k = 1, 2, 3, \dots$$
which is true, since
$$(a+4\sqrt{ab}+b)(a+b) = (a+b)^2+4\sqrt{ab}(a+b) \ge (a+b)^2+4\sqrt{ab}\cdot 2\sqrt{ab} = a^2+10ab+b^2$$
and given $x, y > 0$ we have $x^k + y^k \le (x+y)^k$ for $k = 1, 2, 3, \dots$ □

Solution 2 by Michel Bataille, Rouen, France.
If $a = b$ equality holds, so we suppose that $a < b$ in what follows. Let
$h = 2ab/(a+b), g = \sqrt{ab}, m = (a+b)/2$. We have $h < g < m$ (inequalities of means). The proposed inequality writes as

(1) $$\phi(g) - \phi(h) \le \phi(g+m-h) - \phi(m)$$

where $\phi(x) = e^{x^2}$. Note that $h < g < m < g+m-h$. The Mean Value Theorem show that $\phi(g) - \phi(h) = (g-h)\phi'(u)$ and $\phi(g+m-h) - \phi(m) = (g-h)\phi'(v)$ for some $u \in (h, g), v \in (m, g+m-h)$. An easy calculation gives $\phi''(x) = (4x^2+2)e^{x^2} > 0$, hence ϕ' is an increasing function. Consequently $\phi'(u) < \phi'(v)$ and since $g - h > 0$, (1) follows. □

Solution 3 by Perfetti Paolo, dipatimento di matematica.
Universita di "Tor Vergata", Roma, Italy.
Set
$$\sqrt{ab} =: G, \quad \frac{a+b}{2} =: M, \quad \frac{2ab}{a+b} =: H.$$
We know $M \ge G \ge H$. This inequality in question is equivalent to
$$e^{G^2} - e^{H^2} \le e^{(G+M-H)^2} - e^{M^2}.$$
We observe that $(G+M-H)^2 - M^2 = (G-H)(G+2M-H) \ge 0$. Clearly
$$H^2 \le G^2 \le M^2 \le (G+M-H)^2$$
Lagrange's theorem yields
$$e^{G^2} - e^{H^2} - e^{\xi}(G^2 - H^2) \le e^{(G+M-H)^2} - e^{M^2} = e^{\eta}\Big((G+M-H)^2 - M^2\Big)$$
where $G^2 < \xi < H^2 \le M^2 < \eta < (G+M-H)^2$. Because of $(e^x)^{\eta} = e^x > 0$ we have
$$e^{\eta}\Big((G+M-H)^2 - M^2\Big) \ge e^{\xi}\Big((G+M-H)^2 - M^2\Big)$$

thus it suffices to show that

$$e^{\xi}(G^2-H^2)\le e^{\xi}\Big((G+M-H)^2-M^2\Big)\Leftrightarrow G^2-H^2\le (G+M-H)^2-M^2$$

or

$$GH+MH\le H^2+GM\Leftrightarrow H(M-H)\le G(M-H)$$

and this concludes the proof. □

Solution 4 by proposer.
Lemma:
If $a,b,x,y,z\in\mathbb{R}; a\le x\le y\le z\le b; y+z\le b+x; f:[a,b]\to\mathbb{R}; f$ convexe; then:

$$f(y)+f(z)\le f(x)+f(y+z-x) \tag{1}$$

Proof.

$$x\le z\Rightarrow 0\le z-x\Rightarrow y\le y+z-x$$
$$x\le y\Rightarrow 0\le y-x\Rightarrow z\le y+z-x$$
$$y\in[x;y+z-x]\Rightarrow(\exists)\alpha\in[0,1]$$
$$y=\alpha x+(1-\alpha)(y+z-x) \tag{2}$$
$$z\in[x;y+z-x]\Rightarrow(\exists)\beta\in[0,1]$$
$$z=\beta x+(1-\beta)(y+z-x) \tag{3}$$

By adding $(2);(3)$:

$$y+z=(\alpha+\beta)x+(y+z-x)(2-\alpha-\beta) \tag{4}$$
$$y+z=x+(y+z-x)$$

Replacing $y+z$ in (4):

$$x+(y+z-x)=(\alpha+\beta)x+(y+z-x)(2-\alpha-\beta)$$
$$(\alpha+\beta-1)x+(y+z-x)(1-\alpha-\beta)=0$$
$$(\alpha+\beta-1)(x-y-z+x)=0$$
$$(\alpha+\beta-1)(2x-y-z)=0$$
$$\text{Case I: } 2x-y-z=0\Rightarrow y+z=2x \tag{5}$$
$$\text{But } x\le y; x\le z\Rightarrow 2x\le y+z \tag{6}$$

By $(5);(6)\Rightarrow x=y=z$
Inequality (1) becomes:

$$f(x)+f(x)\le f(x)+f(x+x-x)\text{ (true)}$$
$$\text{Case II: }\alpha+\beta-1=0\Rightarrow\alpha+\beta=1\Rightarrow 2-\alpha-\beta=1$$
$$f\text{ convexe };y,z\in[x,y+z-x]\Rightarrow$$
$$\Rightarrow(\exists)\alpha,\beta\in[0,1]\text{ such that:}$$
$$f(y)\le\alpha f(x)+(1-\alpha)f(y+z-x) \tag{7}$$
$$f(z)\le\beta f(x)+(1-\beta)f(y+z-x) \tag{8}$$

By adding $(7);(8)$:

$$f(y)+f(z)\le(\alpha+\beta)f(x)+(2-\alpha-\beta)f(y+z-x)$$
$$f(y)+f(z)\le f(x)+f(y+z-x)$$

□

Back to main problem:
We take in (1) : $f : (0, \infty) \to \mathbb{R}$;

$$f(x) = e^{x^2}; f'(x) = 2xe^{x^2};$$
$$f''(x) = 2e^{x^2}(1 + 2x^2) > 0; f \text{ convexe.}$$
$$f(y) + f(z) \leq f(x) + f(y + z - x)$$
$$e^{y^2} + e^{z^2} \leq e^{x^2} + e^{(y+z-x)^2} \tag{9}$$

We take in (9):

$$x = \frac{2ab}{a+b}; y = \sqrt{ab}; z = \frac{a+b}{2}$$
$$e^{(\sqrt{ab})^2} + e^{(\frac{a+b}{2})^2} \leq e^{(\frac{2ab}{a+b})^2} + e^{(\sqrt{ab} + \frac{a+b}{2} - \frac{2ab}{a+b})^2}$$
$$e^{ab} + e^{(\frac{a+b}{2})^2} \leq e^{(\frac{2ab}{a+b})^2} + e^{(\sqrt{ab} + \frac{a+b}{2} - \frac{2ab}{a+b})^2}$$

Equality holds for $a = b$. □

5781. Let $m, n, p, q, r, s \in \mathbb{N} \setminus \{0\}$ and define

$$H_n^{(m)} = \frac{1}{1^m} + \frac{1}{2^m} + \ldots + \frac{1}{n^m}.$$

Prove that

$$(H_n^{(2p)} + H_n^{(2q)})(H_n^{(2r)} + H_n^{(2s)}) \geq (H_n^{(p+r)} + H_n^{(q+s)})^2.$$

Daniel Sitaru - Romania

Solution 1 by Michel Bataille, Rouen, France.
Let $a_j = \frac{1}{j^p}$ for $j = 1, 2, \ldots, n$ and $a_j = \frac{1}{(j-n)^q}$ for $j = n+1, n+2, \ldots, 2n$. Similarly, let $b_j = \frac{1}{j^r}$ for $j = 1, 2, \ldots, n$ and $b_j = \frac{1}{(j-n)^s}$ for $j = n+1, n+2, \ldots, 2n$. Then the Cauchy - Schwarz inequality gives

$$\left(\sum_{j=1}^{2n} a_j^2\right)\left(\sum_{j=1}^{2n} b_j^2\right) \geq \left(\sum_{j=1}^{2n} a_j b_J\right)^2,$$

which is nothing else than

$$(H_n^{(2p)} + H_n^{(2q)})(H_n^{(2r)} + H_n^{(2s)}) \geq (H_n^{(p+r)} + H_n^{(q+s)})^2.$$

□

Solution 2 by Perfetti Paolo, dipartimento de matematica.
Universita di "Tor Vergata", Roma, Italy.
Cauchy - Schwarz yields

$$(H_n^{(2p)} + H_n^{(2q)})(H_n^{(2r)} + H_n^{(2s)}) \geq \left(\sqrt{H_n^{(2p)} H_n^{(2r)}} + \sqrt{H_n^{(2q)} H_n^{(2s)}}\right)^2$$

hence we come to

$$\sqrt{H_n^{(2p)} H_n^{(2r)}} + \sqrt{H_n^{(2q)} H_n^{(2s)}} \geq H_n^{(p+r)} + H_n^{(q+s)} \tag{1}$$

By Cauchy - Schwarz again

$$H_n^{(2p)} H_n^{(2r)} = \sum_{k=1}^{n} \frac{1}{k^{2p}} \sum_{k=1}^{n} \frac{1}{k^{2r}} \geq \left(\sum_{k=1}^{n} \frac{1}{k^p} \frac{1}{k^r}\right)^2 = \left(\sum_{k=1}^{n} \frac{1}{k^{r+p}}\right)^2 = (H_n^{(r+p)})^2$$

$$H_n^{(2q)}H_n^{(2s)}=\sum_{k=1}^{n}\frac{1}{k^{2q}}\sum_{k=1}^{n}\frac{1}{k^{2s}}\geq\left(\sum_{k=1}^{n}\frac{1}{k^q}\frac{1}{k^s}\right)^2=\left(\sum_{k=1}^{n}\frac{1}{k^{q+s}}\right)^2=(H_n^{(q+s)})^2$$

and (1) clearly follows. □

Solution 3 by proposer.
By Huygens inequality:

$$\left(H_n^{(2p)}+H_n^{(2q)}\right)\left(H_n^{(2r)}+H_n^{(2s)}\right)\geq$$
$$\geq\left(\sqrt{H_n^{(2p)}\cdot H_n^{(2r)}}+\sqrt{H_n^{(2q)}\cdot H_n^{(2s)}}\right)^2=$$
$$=\left(\sqrt{\left(\sum_{k=1}^{n}\frac{1}{k^{2p}}\right)\left(\sum_{k=1}^{n}\frac{1}{k^{2r}}\right)}+\sqrt{\left(\sum_{k=1}^{n}\frac{1}{k^{2q}}\right)\left(\sum_{k=1}^{n}\frac{1}{k^{2s}}\right)}\right)^2\geq$$
$$\overset{\text{Cauchy-Schwarz}}{\geq}\left(\sqrt{\left(\sum_{k=1}^{n}\frac{1}{k^p}\cdot\frac{1}{k^r}\right)^2}+\sqrt{\left(\sum_{k=1}^{n}\frac{1}{k^q}\cdot\frac{1}{k^s}\right)^2}\right)^2=$$
$$=\left(\sum_{k=1}^{n}\frac{1}{k^{p+r}}+\sum_{k=1}^{n}\frac{1}{k^{q+s}}\right)^2=$$
$$=\left(H_n^{(p+r)}+H_n^{(q+s)}\right)^2$$

□

5789. Let $0<a\leq b$. Suppose $f:[a,b]\to(0,\infty)$ is a continuous function. Then:

$$\int_a^b\int_a^b\int_a^b\left(f^2(x)+f^2(y)+f^2(z)\right)^2dxdydz\geq9(b-a)\left(\int_a^b f(x)dx\right)\left(\int_a^b f^3(x)dx\right).$$

Daniel Sitaru - Romania

Solution 1 by Michel Bataille, Rouen, France.
We use the following lemma: if a,b,c are real numbers, then
$(a^2+b^2+c^2)^2\geq3(a^3b+b^3c+c^3a)$.
Proof (from Z. Cvetkovski, *Inequalities*, Springer, 2012, p. 227)
Let $x=a^2-ab+bc, y=b^2-bc+ca, z=c^2-ca+ab$. The inequality directly follows from the well-known $(x+y+z)^2\geq3(xy+yz+zx)$.
From this lemma, we have

$$(f^2(x)+f^2(y)+f^2(z))^2\geq3(f^3(x)f(y)+f^3(y)f(z)+f^3(z)f(x))$$

for all $(x,y,z)\in[a,b]^3$. Integrating, we obtain

$$\int_a^b\int_a^b\int_a^b(f^2(x)+f^2(y)+f^2(z))^2dxdydz\geq$$
$$3\left((b-a)\int_a^b\int_a^b f^3(x)f(y)dxdy+(b-a)\int_a^b\int_a^b f^3(y)f(z)dydz+(b-a)\int_a^b\int_a^b f^3(z)f(x)dxdz\right)$$

Now, if $I=\int_a^b f^3(x)$ and $J=\int_a^b f(x)dx$, we have

$$\int_a^b\int_a^b f^3(x)f(y)dxdy=\left(\int_a^b f^3(x)dx\right)\left(\int_a^b f(y)dy\right)=I\cdot J$$

and similarly,

$$\int_a^b \int_a^b f^3(y)f(z)dydz = I \cdot J = \int_a^b \int_a^b f^3(z)f(x)dxdz$$

Thus, we have

$$\int_a^b \int_a^b \int_a^b (f^2(x) + f^2(y) + f^2(z))^2 dxdydz \geq 3(3(b-a)I \cdot J) = 9(b-a)IJ,$$

as desired. □

Solution 2 by Perfetti Paolo, dipartimento di matematica. Universita di "Tor Vergata", Roma, Italy.
The inequality is

$$\int_a^b \int_a^b \int_a^b (f^4(x) + f^4(y) + f^4(z) + 2(f(x)f(y))^2 +$$
$$+2(f(y)f(z))^2 + 2(f(x)f(z))^2)dxdydz+$$
$$\geq 9(b-a)\left(\int_a^b f(x)dx\right)\left(\int_a^b f^3(x)dx\right)$$

or

$$3(b-a)^2 \int_a^b f^4(x)dx + 6(b-a)\left(\int_a^b f^2(x)dx\right)\left(\int_a^b f^2(y)dy\right)$$
$$\geq 9(b-a)\left(\int_a^b f(x)dx\right)\left(\int_a^b f^3(x)dx\right)$$
$$(b-a)\int_a^b f^4(x)dx + 2\left(\int_a^b f^2(x)dx\right)\left(\int_a^b f^2(y)dy\right) \geq$$
$$\geq 3\left(\int_a^b f(x)dx\right)\left(\int_a^b f^3(x)dx\right)$$

This may be rewritten as

$$\int_a^b \int_a^b f^4(x)dxdy + 2\int_a^b \int_a^b f^2(x)f^2(y)dxdy \geq 3\int_a^b \int_a^b f(x)f^3(y)dxdy$$

that is

$$\int_a^b \int_a^b (f^4(x) + 2f^2(x)f^2(y) - 3f(x)f^3(y))dxdy$$

or

$$\int_a^b \int_a^b f(x)(f(x) - f(y))(f^2(x) + f(x)f(y) + 3f^2(y))dxdy$$

Now let's consider the two points (x,y) and (y,x). The sum of the two terms are

$$f(x)f(x) - f(y))(f^2(x) + f(x)f(y) + 3f^2(y))+$$
$$+f(y)(f(y) - f(x))(f^2(y) + f(y)f(x) + 3f^2(x)) =$$
$$= (f(x) - f(y))^2(f^2(x) + f^2(y) + 3f(x)f(y)) \geq 0$$

thus concluding the proof. □

Solution 3 by proposer.
Lemma: If $a, b, c \in \mathbb{R}$ then:

$$(a^2+b^2+c^2)^2 \geq 3(a^3b+b^3c+c^3a)$$

Proof.

$$0 \leq \sum_{cyc}(a^2-b^2-ab-ac+2bc)^2 =$$

$$= \sum_{cyc} a^4 + \sum_{cyc} b^4 + \sum_{cyc} a^2b^2 + \sum_{cyc} a^2c^2 + 4\sum_{cyc} b^2c^2 -$$

$$-2\sum_{cyc} a^2b^2 - 2\sum_{cyc} a^3b - 2\sum_{cyc} a^3c + 4abc\sum_{cyc} a +$$

$$+2\sum_{cyc} ab^3 + 2abc\sum_{cyc} b - 4\sum_{cyc} b^3c + 2abc\sum_{cyc} c -$$

$$-4abc\sum_{cyc} b - 4abc\sum_{cyc} c =$$

$$= 2\sum_{cyc} a^4 + 4\sum_{cyc} b^2c^2 - 6\sum_{cyc} a^3b =$$

$$= 2(a^2+b^2+c^2)^2 - 6(a^3b+b^3c+c^3a)$$

$$0 \leq 2(a^2+b^2+c^2)^2 - 6(a^3b+b^3c+c^3a)$$

$$(a^2+b^2+c^2)^2 \geq 3(a^3b+b^3c+c^3a) \tag{1}$$

Let's take in $(1): a = f(x); b = f(y); c = f(z)$

$$(f^2(x)+f^2(y)+f^2(z))^2 \geq 3(f^3(x)f(y)+f^3(y)f(z)+f^3(z)f(x))$$

$$\int_a^b\int_a^b\int_a^b (f^2(x)+f^2(y)+f^2(z))^2 dxdydz \geq$$

$$\geq 3\sum_{cyc}\int_a^b\int_a^b\int_a^b f^3(x)f(y)dxdydz =$$

$$= 3\sum_{cyc}\left(\int_a^b f^3(x)dx\right)\left(\int_a^b f(y)dy\right)\left(\int_a^b dz\right) =$$

$$= 3(b-a)\sum_{cyc}\left(\int_a^b f^3(x)dx\right)\left(\int_a^b f(x)dx\right) =$$

$$= 9(b-a)\left(\int_a^b f(x)dx\right)\left(\int_a^b f^3(x)dx\right)$$

Equality holds for $a = b$ or $f \equiv 1$. □

□

5793. Suppose $f : [a, b] \to [1, \infty)$ is a continuous function with $0 < a \leq b$. Then:

$$n(b-a)^{n-1}\int_a^b f(x)dx \leq (n-1)(b-a)^n + \left(\int_a^b f(x)dx\right)^n.$$

Daniel Sitaru - Romania

Solution 1 by Ángel Plaza, Universidad de Las Palmas de Gran Canaria, Spain.
The proposed inequality may be written as

$$\frac{\int_a^b f(x)dx}{b-a} \le \frac{(n-1) + \left(\frac{\int_a^b f(x)dx}{b-a}\right)^n}{n},$$

which follows by the AM-GM inequality. □

Solution 2 by Brian Bradie, Department of Mahtematics. Christopher Newport University, Newport News, VA.
Let x and y be non-negative real numbers. By the arithmetic mean - geometric mean inequality,

$$(n-1)x + y = \underbrace{x + x + \ldots + x}_{n-1 \text{ terms}} + y \ge n\sqrt[n]{x^{n-1}y}. \tag{1}$$

Because $f : [a,b] \to [1,\infty)$ is a continuous function and $a \le b$,

$$b - a \ge 0 \qquad \text{and} \qquad \int_a^b f(x)dx \ge 0.$$

Substituting

$$x = (b-a)^n \qquad \text{and} \qquad y = \left(\int_a^b f(x)dx\right)^n$$

into (1) then yields

$$(n-1)(b-a)^n + \left(\int_a^b f(x)dx\right)^n \ge n\sqrt[n]{(b-a)^{n(n-1)}\left(\int_a^b f(x)dx\right)^n}$$

$$= n(b-a)^{n-1}\int_a^b f(x)dx.$$

□

Solution 3 by David A. Huckaby, Angelo State University, San Angelo, TX.
If $a = b$, both sides of the inequality are zero, so we assume that $a < b$. Dividing both sides by $(b-a)^n$, the desired inequality is

$$n \cdot \frac{1}{b-a}\int_a^b f(x)dx \le n - 1 + \left(\frac{1}{b-a}\int_a^b f(x)dx\right)^n.$$

We first note that this inequality does not holds for all real n. For example, letting $a = 0, b = 1, f(x) = 4$, and $n = \frac{1}{2}$, we have $\frac{1}{2}(4) \not\le \frac{1}{2} - 1 + (4)^{\frac{1}{2}}$. We will show that the inequality holds for $n \ge 1$ and for $n \le 0$. That the inequality holds for $n = 0$ and for $n = 1$ can be seen be inspection, so we will consider the cases $n > 1$ and $n < 0$.
By the Mean Value Theorem for Integrals, there is a $c \in [a,b]$ such that
$f(c) = \int_a^b f(x)dx$, so that the desired inequality is $nf(c) \le n - 1 + [f(c)]^n$, that is, $n(f(c) - 1) \le [f(c)]^n - 1$. If $f(c) = 1$, then both sides of this inequality are zero. Otherwise, $f(c) > 1$ by assumption, and we consider the equivalent inequality

$$n \le \frac{[f(c)]^n - 1}{f(c) - 1}. \tag{2}$$

Let $g(x) = \frac{x^n-1}{x-1}$ for $x > 1$. by l'Hôpital's Rule we have

$$\lim_{x\to 1^+} g(x) = \lim_{x\to 1^+} \frac{x^n-1}{x-1} = \lim_{x\to 1^+} \frac{nx^{n-1}}{1} = n.$$

So if we show that $g(x)$ is an increasing function for $x > 1$, then inequality (2) follows. To this end we note that $g'(x) = \frac{(n-1)x^{n+1}-nx^n+x}{(x-1)^2x}$.

For the case $n > 1$, we rewrite $g'(x) = \frac{(n-1)x^n-nx^{n-1}+1}{(x-1)^2}$. The denominator is clearly positive. Denoting the numerator by $h(x) = (n-1)x^n - nx^{n-1} + 1$, we have $h'(x) = n(n-1)x^{n-1} - n(n-1)x^{n-2} = n(n-1)(x^{n-1} - x^{n-2}) > 0$. Since $h(1) = 0$, this shows that $h(x) > 0$ for $x > 1$,

so that $g'(x) > 0$ for $x > 1$. Thus inequality (2) hols for $n \geq 1$.

(When $n > 1$ is an integer, a shorter route is to note that inequality (2) is simply $n \leq \frac{(f(c)-1)([f(c)]^{n-1}+[f(c)]}{f(c)-1}$ that is, $n \leq [f(c)]^{n-1} + [f(c))]^{n-2} + \ldots + f(c) + 1$, which is clearly true for $f(c) > 1$.)

For the case $n < 0$, we note that the denominator of $g'(x) = \frac{(n-1)x^{n+1}-nx^n+x}{(x-1)^2x}$ is positive. Denoting the numerator by $k(x) = (n-1)x^{n+1} - nx^n + x$, we have

$k'(x) = (n+1)(n-1)x^n - n^2x^{n-1} + 1 = (n^2-1)x^n - n^2x^{n-1} + 1 = n^2(x^n - x^{n-1}) - x^n + 1 > 0$, since $x^n < 1$ for $n < 0$ and $x > 1$. Since $k(1) = 0$, this shows that $k(x) > 0$ for $x > 1$, so that $g'(x) > 0$ for $x > 1$. Thus inequality (2) also holds for $n < 0$.

So the original inequality holds for $n \geq 1$ and for $n \leq 0$.

Let $I = \int_a^b f(x)dx$. Since $b - a \geq 0$ and $I \geq 0$, we can apply the arithmetic mean - geometric mean inequality as follows:

$$(n-1)(b-a)^n + I^n = (b-a)^n + \ldots + (b-a)^n + I^n \geq n((b-a)^n \ldots (b-a)^n \cdot I^n)^{\frac{1}{n}}$$

and deduce that

$$(n-1)(b-a)^n + I^n \geq n((b-a)^{n(n-1)} \cdot I^n)^{\frac{1}{n}} = n(b-a)^{n-1}I,$$

as desired. □

Solution 5 by Perfetti Paolo, dipartimento di matematica Universita di. "Tor Vergata", Roma, Italy.

$$\left(\int_a^b f(x)dx\right)^n + \underbrace{(b-a)^n + \ldots (b-a)^n}_{n-1\ times} \geq \left(\left(\int_a^b f(x)dx\right)^n (b-a)^{n(n-1)}\right)^{\frac{1}{n}} =$$

$$= n(b-a)^{n-1} \int_a^b f(x)dx$$

□

Solution by Ulrich Abel, Technische Hochschule Mittelhessen, Friedberg, Germany.

As $f(x) \geq 1$, we have $\int_a^b f(x)dx = (b-a)(1+y)$ with a real number $y \geq 0$. We have to show that

$$n(b-a)^n(1+y) \leq (n-1)(b-a)^n + (b-a)^n(1+y)^n.$$

If $a = b$ the inequality is obvious. In case $a < b$ the inequality is equivalent to

$$n(1+y) \leq n-1 + (1+y)^n$$

or, after an elementary simplification, to

$$1 + ny \leq (1 + y)^n.$$

This is just the Bernoulli inequality. □

Solution 7 by Albert Stadler, Herrliberg, Switzerland.
We assume that n is a real variable with $n \geq 1$. The inequality holds trivially true if $b = a$. Let $b > a$, and let $I := \frac{1}{b-a}\int_a^b f(x)dx$. The inequality then reads as

$$nI \leq n - 1 + I^n$$

which is exactly Bernoulli's inequality
(see for instance https://en.wikipedia.org/wiki/Bernoulli%27s_inequality):

$$(1 + x)^r \geq 1 + rx$$

for every real number $r \geq 1$ and $x \geq -1$. The inequality is strict if $x \neq 0$ and $r \neq 1$. Hence the asumption that $f(x) \geq 1$ is not requiered. It suffices to assume that f is nonnegative. □

Solution 8 by proposer.
We prove by induction that:

$$x_1, x_2, \ldots, x_n \in [1, \infty); n \in \mathbb{N}^* \text{ implies:}$$

$$x_1 + x_2 + \ldots + x_n \leq n - 1 + x_1 x_2 \ldots x_n$$

$$\text{Checking: } n = 1; x_1 = 1 - 1 + x_1 \Leftrightarrow x_1 \leq x_1$$

$$n = 2 : x_1 + x_2 \leq 1 + x_1 x_2 \Leftrightarrow (x_1 - 1)(x_2 - 1) \geq 0. \text{ True.}$$

$$P(k) : x_1 + x_2 + \ldots + x_k \leq k - 1 + x_1 x_2 \ldots x_k \tag{1}$$

Suppose that it's true.

$$P(k+1) : x_1 + x_2 + \ldots + x_k + x_{k+1} \leq k + x_1 x_2 \ldots x_k x_{k+1} \text{ (to prove)}$$

$$x_1 + x_2 + \ldots + x_k + x_{k+1} \overset{P(k)}{\leq} k - 1 + x_1 x_2 \ldots x_k + x_{k+1}$$

Remains to prove that:

$$k - 1 + x_1 x_2 \ldots x_k + x_{k+1} \leq k + x_1 x_2 \ldots x_k x_{k+1}$$

$$x_1 x_2 \ldots x_k x_{k+1} - x_1 x_2 \ldots x_k - x_{k+1} + 1 \geq 0$$

$$x_1 x_2 \ldots x_k (x_{k+1} - 1) - (x_{k+1} - 1) \geq 0$$

$$(x_{k+1} - 1)(x_1 x_2 \ldots x_k - 1) \geq 0 \text{ which is true because}$$

$$x_{k+1} \geq 1; x_1 x_2 \ldots x_k \geq 1$$

$$P(k) \to P(k+1)$$

In (1) we take:

$$x_1 \to f(x_1); x_2 \to f(x_2); \ldots; x_n \to f(x_n)$$

$$\sum_{k=1}^{n} f(x_k) \leq n - 1 + \prod_{k=1}^{n} f(x_k) \tag{2}$$

By integration in (2):

$$\int_a^b \int_a^b \cdot \ldots \cdot \int_a^b \left(\sum_{k=1}^{n} f(x_k)\right) dx_1 dx_2 \cdot \ldots \cdot dx_n \leq$$

$$\leq \int_a^b \int_a^b \cdot \ldots \cdot \int_a^b (n-1)dx_1dx_2\ldots dx_n + \int_a^b \int_a^b \cdot \ldots \cdot \int_a^b \prod_{k=1}^n f(x_k)dx_1dx_2\ldots dx_n$$

$$\sum_{k=1}^n \int_a^b \int_a^b \cdot \ldots \cdot \int_a^b f(x_k)dx_1dx_2\ldots dx_n \leq$$

$$\leq (n-1)(b-a)^n + \prod_{k=1}^n \int_a^b f(x_k)dx_k$$

$$n(b-a)^{n-1}\int_a^b f(x)dx \leq (n-1)(b-a)^n + \left(\int_a^b f(x_k)dx_k\right)^n$$

Equality holds for $a = b$ or $f(x) \equiv 1$. □

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