ROMANIAN MATHEMATICAL MAGAZINE

Find:

$$\Omega = \lim_{n\to\infty}\left[\frac{1}{n}\int_0^{\frac{\pi}{4}}\ln(1+e^{ncosx})\,dx\right]$$

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$$\Omega = \lim_{n\to\infty}\left[\frac{1}{n}\int_0^{\frac{\pi}{4}}\ln(e^{ncosx}(1+e^{-ncosx})dx\right] =$$

$$= \lim_{n\to\infty}\frac{1}{n}\left(\int_0^{\frac{\pi}{4}}\ln(e^{ncosx})\,dx + \int_0^{\frac{\pi}{4}}\ln(1+e^{-ncosx})\,dx\right) =$$

$$= \int_0^{\frac{\pi}{4}}\cos(x)\,dx + \lim_{n\to\infty}\frac{1}{n}\int_0^{\frac{\pi}{4}}\ln(1+e^{-ncosx})\,dx = [\sin(x)]_0^{\frac{\pi}{4}} + L =$$

$$= \frac{\sqrt{2}}{2} + L \rightarrow \quad L = \lim_{n\to\infty}\frac{1}{n}\int_0^{\frac{\pi}{4}}\ln(1+e^{-ncosx})\,dx \rightarrow \ f_n(x) = \frac{\ln(1+e^{-ncosx})}{n}$$

Lebesgue's Dominated Convergence Theorem:

$$\ln(1+t) \le t \rightarrow 0 \le \frac{\ln(1+e^{-ncosx})}{n} \le \frac{e^{-ncosx}}{n}$$

$$n \ge 1, x\epsilon\left[0;\frac{\pi}{4}\right] \rightarrow \quad e^{-ncosx} \le 1$$

$$\frac{\ln(1+e^{-ncosx})}{n} \le \frac{1}{n} \le 1 \rightarrow \quad L = \lim_{n\to\infty}\int_0^{\frac{\pi}{4}}f_n(x)dx = \int_0^{\frac{\pi}{4}}(\lim_{n\to\infty}f_n(x))dx$$

$$\lim_{n\to\infty}f_n(x) = 0 \rightarrow \quad L = 0$$

$$\Omega = \frac{\sqrt{2}}{2} + L = \frac{\sqrt{2}}{2}$$