ROMANIAN MATHEMATICAL MAGAZINE

If $0 < a \le b$ then:

$$\int_a^b \int_a^b \frac{dxdy}{(2+x+y)^2} \le \frac{b-a}{4}\left(\frac{1}{1+a} - \frac{1}{1+b}\right)$$

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$Lemma : \forall\, x > -1, \ln(1+x) \le x$

$Proof: Let\ f(x) = x - \ln(1+x)\ then\ f'(x) = 1 - \frac{1}{1+x} = \frac{x}{1+x}$

$clearly\ for\ x \in (-1,0), f'(x) < 0\ so\ f(x) decreasing\ and$

$x \in (0,\infty), f'(x) > 0\ so\ f(x) increasing$

$so\ f(x)\ has\ a\ global\ minimum\ at\ x = 0\ and\ f(0) = 0$

$so\ f(x) \ge f(0) \Rightarrow\ \ln(1+x) \le x$

$$\int_a^b \int_a^b \frac{dxdy}{(2+x+y)^2} = \int_a^b \left(\int_a^b \frac{1}{(2+x+y)^2}\right) dy = \int_a^b \left(-\frac{1}{2+x+y}\right)_a^b dy =$$

$$= \int_a^b \left(\frac{1}{2+a+y} - \frac{1}{2+b+y}\right) dy = \left[\ln\frac{2+a+y}{2+b+y}\right]_a^b = \ln\frac{(2+a+b)^2}{4(1+a)(1+b)} =$$

$$= \ln\left(1 + \frac{(b-a)^2}{4(1+a)(1+b)}\right) \overset{lemma}{\le} \frac{(b-a)^2}{4(1+a)(1+b)} = \frac{b-a}{4} \cdot \frac{b-a}{(1+a)(1+b)} =$$

$$= \frac{b-a}{4} \cdot \frac{(1+b)-(1+a)}{(1+b)(1+a)} = \frac{b-a}{4}\left(\frac{1}{1+a} - \frac{1}{1+b}\right)$$

Equality holds for a=b.