ROMANIAN MATHEMATICAL MAGAZINE

Find:

$$\Omega=\int_0^\infty \frac{x\,ln^2(x)}{(1+x^2)^2}dx$$

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$$\Omega=\int_0^\infty \frac{x\,ln^2(x)}{(1+x^2)^2}dx\quad x^2=t,\qquad \Omega=\frac{1}{8}\int_0^\infty \frac{ln^2(t)}{(1+t)^2}dt=$$

$$=\frac{1}{8}\lim_{a\to 0}\frac{d^2}{da^2}\int_0^\infty \frac{t^a}{(1+t)^2}dt=\frac{1}{8}\lim_{a\to 0}\frac{d^2}{da^2}\int_0^\infty \frac{t^{(a+1)-1}}{(1+t)^{(1+a)+(1-a)}}dt=$$

$$=\frac{1}{8}\lim_{a\to 0}\frac{d^2}{da^2}\beta(a+1;1-a)=\frac{1}{8}\lim_{a\to 0}\frac{d^2}{da^2}\Gamma(1-a)\Gamma(1+a)=$$

$$=\frac{1}{8}\lim_{a\to 0}\Gamma(1-a)\Gamma(1+a)(\left(\psi^{(0)}(1-a)-\psi^{(0)}(1+a)\right)^2+\ \psi^{(1)}(1-a)+\psi(1+a))=$$

$$=\frac{1}{8}\left(2\psi^{(1)}(1)\right)=\frac{\pi^2}{24}$$

$$\int_0^\infty \frac{x\ln^2(x)}{(1+x^2)^2}dx=\frac{\pi^2}{24}$$