ROMANIAN MATHEMATICAL MAGAZINE

Find:

$$I=\int_0^\infty \frac{dx}{1+x^6}$$

Proposed by Nguyen Hung Cuong-Vietnam

Solution 1 by Togrul Ehmedov-Azerbaijan

$$\text{We know that the equality} \int_0^\infty \frac{dx}{1+x^k}=\frac{\pi}{k\sin\left(\frac{\pi}{k}\right)} \text{ is ture. Now let's prove it.}$$

$$\text{Let } x^k=y \Rightarrow dx=\frac{1}{k}y^{\frac{1-k}{k}}dy$$

$$I=\int_0^\infty \frac{dx}{1+x^k}=\frac{1}{k}\int_0^\infty \frac{y^{\frac{1-k}{k}}}{1+y}dy$$

$$\text{Let} \frac{1}{1+y}=1-t \Rightarrow dy=\frac{dt}{(1-t)^2}$$

$$I=\frac{1}{k}\int_0^1 t^{\left(\frac{1}{k}-1\right)}(1-t)^{-\frac{1}{k}}dt=\frac{1}{k}B\left(\frac{1}{k},1-\frac{1}{k}\right)=\frac{1}{k}\Gamma\left(\frac{1}{k}\right)\Gamma\left(1-\frac{1}{k}\right)=\frac{\pi}{k\sin\left(\frac{\pi}{k}\right)}$$

$$I=\int_0^\infty \frac{dx}{1+x^6}=\frac{\pi}{6\sin\left(\frac{\pi}{6}\right)}=\frac{\pi}{3}$$

Solution 2 by Daniel Sitaru-Romania

$$I=\int_0^\infty \frac{dx}{1+x^6}\overset{y=x^6}{\cong}\int_0^\infty \frac{1}{1+y}\cdot\frac{1}{6\sqrt[6]{y^5}}dy=\frac{1}{6}\int_0^\infty \frac{y^{-\frac{5}{6}}}{1+y}dy=$$

$$=\frac{1}{6}\int_0^\infty \frac{y^{\frac{1}{6}-1}}{(1+y)^{\frac{1}{6}+\frac{5}{6}}}dy=\frac{1}{6}B\left(\frac{1}{6},\frac{5}{6}\right)=\frac{1}{6}\cdot\frac{\Gamma\left(\frac{1}{6}\right)\Gamma\left(\frac{5}{6}\right)}{\Gamma\left(\frac{1}{6}+\frac{5}{6}\right)}=$$

$$=\frac{1}{6}\cdot\frac{\Gamma\left(\frac{1}{6}\right)\Gamma\left(1-\frac{1}{6}\right)}{\Gamma(1)}=\frac{1}{6}\cdot\frac{\pi}{\sin\frac{\pi}{6}}=\frac{\pi}{6\cdot\frac{1}{2}}=\frac{\pi}{3}$$