ROMANIAN MATHEMATICAL MAGAZINE

Find:

$$\Omega = \int_0^\infty \frac{dx}{\sqrt{1+x^8}}$$

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$$\Omega = \int_0^\infty \frac{dx}{\sqrt{1+x^8}} \overset{y=x^8}{\cong} \int_0^\infty \frac{1}{\sqrt{1+y}} \cdot \frac{1}{8\sqrt[8]{y^7}} dy = \frac{1}{8}\int_0^\infty \frac{y^{-\frac{7}{8}}}{(1+y)^{\frac{1}{2}}} dy =$$

$$= \frac{1}{8}\int_0^\infty \frac{y^{\frac{1}{8}-1}}{(1+y)^{\frac{1}{8}+\frac{3}{8}}} dy = \frac{1}{8} B\left(\frac{1}{8}, \frac{3}{8}\right) = \frac{1}{8} \cdot \frac{\Gamma\left(\frac{1}{8}\right)\Gamma\left(\frac{3}{8}\right)}{\Gamma\left(\frac{1}{8}+\frac{3}{8}\right)} = \frac{1}{8} \cdot \frac{\Gamma\left(\frac{1}{8}\right)\Gamma\left(\frac{3}{8}\right)}{\Gamma\left(\frac{1}{2}\right)} = \frac{\Gamma\left(\frac{1}{8}\right)\Gamma\left(\frac{3}{8}\right)}{8\sqrt{\pi}}$$

Observation:

$$\Gamma\left(\frac{1}{2}\right) \cdot \Gamma\left(1-\frac{1}{2}\right) = \frac{\pi}{\sin\frac{\pi}{2}}$$

$$\left(\Gamma\left(\frac{1}{2}\right)\right)^2 = \pi \Rightarrow \Gamma\left(\frac{1}{2}\right) = \sqrt{\pi}$$