ROMANIAN MATHEMATICAL MAGAZINE

$$\textbf{Find:}$$

$$I=\int_{\frac{\pi}{6}}^{\frac{\pi}{3}}\frac{sinx}{(1+cotx)^2}dx$$

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$$Rule:\int_a^b f(x)dx=\int_a^b f(a+b-x)dx\ (1)$$

$$a^3+b^3=(a+b)(a^2-ab+b^2)$$

$$Let\ I=\int_{\frac{\pi}{6}}^{\frac{\pi}{3}}\frac{sinx}{(1+cotx)^2}dx=\int_{\frac{\pi}{6}}^{\frac{\pi}{3}}\frac{\sin^3 x}{(sinx+cosx)^2}dx\ \ (1)$$

$$I\overset{(1)}{=}\int_{\frac{\pi}{6}}^{\frac{\pi}{3}}\frac{\sin^3\left(\frac{\pi}{2}-x\right)}{\left(\sin\left(\frac{\pi}{2}-x\right)+\cos\left(\frac{\pi}{2}-x\right)\right)^2}dx\ =\int_{\frac{\pi}{6}}^{\frac{\pi}{3}}\frac{\cos^3 x}{(sinx+cosx)^2}dx\ \ (2)$$

$$adding\ (1)\ \&\ (2) we\ get\ ,$$

$$2I=\int_{\frac{\pi}{6}}^{\frac{\pi}{3}}\frac{\sin^3 x+.\cos^3 x}{(sinx+cosx)^2}dx\overset{(2)}{=}\int_{\frac{\pi}{6}}^{\frac{\pi}{3}}\frac{(1-sinxcosx)}{(sinx+cosx)}dx=\frac{1}{2}\int_{\frac{\pi}{6}}^{\frac{\pi}{3}}\frac{(2-2sinxcosx)}{(sinx+cosx)}dx$$

$$=\frac{1}{2}\int_{\frac{\pi}{6}}^{\frac{\pi}{3}}\frac{(2-sin2x)}{\sqrt{(sinx+cosx)^2}}dx=\frac{1}{2}\int_{\frac{\pi}{6}}^{\frac{\pi}{3}}\frac{3-(1+sin2x)}{\sqrt{1+sin2x}}dx=$$

$$=\frac{3}{2}\int_{\frac{\pi}{6}}^{\frac{\pi}{3}}\frac{1}{\sqrt{1+sin2x}}dx-\frac{1}{2}\int_{\frac{\pi}{6}}^{\frac{\pi}{3}}\sqrt{1+sin2x}dx=$$

$$=\frac{3}{2}\int_{\frac{\pi}{6}}^{\frac{\pi}{3}}\frac{1}{sinx+cosx}dx-\frac{1}{2}\int_{\frac{\pi}{6}}^{\frac{\pi}{3}}(sinx+cosx)dx=$$

$$= \frac{3}{2\sqrt{2}} \int_{\frac{\pi}{6}}^{\frac{\pi}{3}} \frac{1}{\frac{1}{\sqrt{2}} sinx + \frac{1}{\sqrt{2}} cosx} dx - \frac{1}{2}(-cosx + sinx)_{\frac{\pi}{6}}^{\frac{\pi}{3}} =$$

$$= \frac{3}{2\sqrt{2}} \int_{\frac{\pi}{6}}^{\frac{\pi}{3}} \frac{1}{\sin\left(\frac{\pi}{4} + x\right)} dx - \frac{1}{2}(\sqrt{3} - 1) = \frac{3}{2\sqrt{2}} \int_{\frac{\pi}{6}}^{\frac{\pi}{3}} \csc\left(x + \frac{\pi}{4}\right) dx - \frac{1}{2}(\sqrt{3} - 1) =$$

$$= \frac{3}{2\sqrt{2}} \left(\log\left|\tan\left(\frac{x}{2} + \frac{\pi}{8}\right)\right|\right)_{\frac{\pi}{6}}^{\frac{\pi}{3}} - \frac{1}{2}(\sqrt{3} - 1) = \frac{3}{2\sqrt{2}} \log\left|\frac{\tan\frac{7\pi}{24}}{\tan\frac{5\pi}{24}}\right| - \frac{1}{2}(\sqrt{3} - 1)$$

$$So\ 2I = \frac{3}{2\sqrt{2}} \log\left|\frac{\tan\frac{7\pi}{24}}{\tan\frac{5\pi}{24}}\right| - \frac{1}{2}(\sqrt{3} - 1)\ \ or\ I = \frac{1}{2}\left(\frac{3}{2\sqrt{2}} \log\left|\frac{\tan\frac{7\pi}{24}}{\tan\frac{5\pi}{24}}\right| - \frac{1}{2}(\sqrt{3} - 1)\right)$$