ROMANIAN MATHEMATICAL MAGAZINE

Prove that:

$$I=\int_0^\infty \frac{2\pi t}{\ln^2(x)+4\pi^2t^2}\cdot\frac{dx}{1+x^2}=\frac{1}{2}\left(\psi\left(t+\frac{3}{4}\right)-\psi\left(t+\frac{1}{4}\right)\right)$$

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Solution 1 by Amin Hajiyev-Azerbaijan

$$Laplace\ transform\ \ L_x\{\sin(bx)\}(s)=\frac{b}{s^2+b^2}\quad s=|\ln(x)|,b=2\pi t$$

$$Laplace-Fourier\ integral\ \int_0^\infty e^{u|\ln(x)|}\sin(2\pi tu)\,du=\frac{2\pi t}{|\ln(x)|^2+(2\pi t)^2}.$$

$$I=\int_0^\infty\left(\int_0^\infty x^u\sin(2\pi tu)\,du\right)\cdot\frac{1}{1+x^2}dx=\int_0^\infty \sin(2\pi tu)(\int_0^\infty\frac{x^u}{1+x^2}dx)du$$

$$I_u=\int_0^\infty\frac{x^u}{1+x^2}dx=\frac{1}{2}\int_0^\infty\frac{x^{\frac{u}{2}+\frac{1}{2}-1}}{(1+x)^{\left(\frac{u}{2}+\frac{1}{2}\right)+\left(\frac{1}{2}-\frac{u}{2}\right)}}dx=\frac{1}{2}\beta\left(\frac{u+1}{2};\frac{1-u}{2}\right)=$$

$$=\frac{1}{2}\Gamma\left(1-\frac{u}{2}-\frac{1}{2}\right)\Gamma\left(\frac{u}{2}+\frac{1}{2}\right)$$

$$Gamma\ reflection\ formula:\Gamma(1-z)\Gamma(z)=\frac{\pi}{\sin(\pi z)}$$

$$I_u=\frac{\pi}{2\cos\left(\pi\left(\frac{u}{2}+\frac{1}{2}\right)\right)}=\frac{\pi}{2\cos\left(\frac{\pi u}{2}\right)}\to\quad I=\frac{\pi}{2}\int_0^\infty\frac{\sin(2\pi tu)}{\cos\left(\frac{\pi u}{2}\right)}\mathrm{du}$$

$$\frac{1}{\cos\left(\frac{\pi u}{2}\right)}=\frac{2e^{-\frac{\pi u}{2}}}{1+e^{-\pi u}}=2\sum_{n=0}^\infty(-1)^n e^{\left(\left(n+\frac{1}{2}\right)\right)\pi u}\ \to\ I$$

$$=\pi\sum_{n=0}^\infty(-1)^n\int_0^\infty e^{-\left(n+\frac{1}{2}\right)\pi u}\sin(2\pi tu)\,du$$

$$Sine\ integral:\ \int_0^\infty e^{-au}\sin(bu)du=\frac{b}{a^2+b^2}$$

$$I=\pi\sum_{n=0}^\infty(-1)^n\frac{2\pi t}{\pi^2\left(n+\frac{1}{2}\right)^2+4\pi^2t^2}=2t\sum_{n=0}^\infty\frac{(-1)^n}{\left(n+\frac{1}{2}\right)^2+(2t)^2}$$

$$Applying\ the\ Mittag-Leffler\ pole\ expansion\ to\ convert\ the$$
$$trigonometric\ integrand\ into\ a\ Digamma-related\ series.$$

$$\rightarrow I=\sum_{n=0}^{\infty}\frac{(-1)^n}{2t+n+\frac{1}{2}}\quad n=2k, n=2k+1$$

$$I=\sum_{k=0}^{\infty}\left(\frac{1}{2t+2k+\frac{1}{2}}-\frac{1}{2t+2k+\frac{3}{2}}\right)=\frac{1}{2}\sum_{k=0}^{\infty}\left(\frac{1}{k+t+\frac{1}{4}}-\frac{1}{k+t+\frac{3}{4}}\right)$$

$$Digamma\ function: \psi(z+a)-\psi(z+b)=\sum_{k=0}^{\infty}\left(\frac{1}{k+z+b}-\frac{1}{k+z+a}\right)$$

$$\int_0^{\infty}\frac{2\pi t}{\ln^2(x)+4\pi^2t^2}\cdot\frac{1}{1+x^2}dx=\frac{1}{2}\left(\psi\left(t+\frac{3}{4}\right)-\psi\left(t+\frac{1}{4}\right)\right)$$

Solution 2 by Amin Hajiyev-Azerbaijan

$I=\int_0^{\infty}\frac{2\pi t}{ln^2(x)+4\pi^2t^2}\cdot\frac{dx}{1+x^2}$ substitution: $e^u=x,\ dx=e^u du$

$$I=\int_{-\infty}^{\infty}\frac{2\pi t}{u^2+(2\pi t)^2}\cdot\frac{e^u}{e^{2u}+1}du,\qquad \cosh(u)=\frac{e^u+e^{-u}}{2}=\frac{e^{2u}+1}{2e^u}$$

$$I=\pi t\int_{-\infty}^{\infty}\frac{1}{((u^2+(2\pi t)^2)\cosh(u))}du,\ \ f(z)=\frac{1}{(z^2+(2\pi t)^2)\cosh(z)}\ \{\mathrm{Im}\{z\}>0\}$$

$$a)\ z^2+(2\pi t)^2=0\rightarrow\ z=2\pi it\ b)\ \cosh(z)=0\rightarrow\ z_k=i\pi\left(k+\frac{1}{2}\right)\ \ k\epsilon Z^+$$

$$a)\ z=2\pi it\rightarrow\ Res(f,2\pi it)=\lim_{z\to2\pi it}\frac{z-2\pi it}{(z-2\pi it)(z+2\pi it)\cosh(z)}=\frac{1}{4\pi it\cos(2\pi t)}$$

$$b)\ z_k=i\pi\left(k+\frac{1}{2}\right)\rightarrow\left\{Res(f,z_0)=\frac{P(z_0)}{Q'(z_0)}\right\}\ Res(f,z_k)=\frac{1}{(z_k^2+(2\pi t)^2)\sinh(z_k)}=$$

$$=\frac{1}{\left(-\pi^2\left(k+\frac{1}{2}\right)^2+(2\pi t)^2\right)\sinh\left(\pi i\left(k+\frac{1}{2}\right)\right)}=\frac{1}{\pi^2 i\left((2t)^2-\left(k+\frac{1}{2}\right)^2\right)\sin\left(\frac{\pi}{2}+\pi k\right)}$$

Note: $\sinh(ia)=i\sin(a)$,

$$\sin\left(\frac{\pi}{2}+\pi k\right)=(-1)^k\ k\epsilon Z^+\ \text{->}\ Res(f,z_k)=\frac{i(-1)^k}{\pi^2\left(\left(k+\frac{1}{2}\right)^2-(2\pi t)^2\right)}$$

$$I=2\pi i\sum Res=2\pi^2ti\left(Res(f,2\pi it)+Res(f,z_k)\right)=$$

$$=\frac{\pi}{2t\cos(2\pi t)}+2t\sum_{k=0}^{\infty}\frac{(-1)^k}{4t^2-\left(k+\frac{1}{2}\right)^2}$$

$$\bullet\, Theorem\ Mittag-Leffler$$

$$\frac{\pi}{2\cos(2\pi t)}=\sum_{k=-\infty}^{\infty}(-1)^k\left[\frac{1}{k-2t+\frac{1}{2}}\right]==\frac{1}{2}\sum_{k=0}^{\infty}(-1)^k\left[\frac{1}{k-2t+\frac{1}{2}}+\frac{1}{k+2t+\frac{1}{2}}\right]$$

$$I=\frac{1}{2}\sum_{k=0}^{\infty}(-1)^k\left(\frac{1}{k+\frac{1}{2}-2t}+\frac{1}{k+\frac{1}{2}+2t}-\frac{1}{k+\frac{1}{2}-2t}+\frac{1}{k+\frac{1}{2}+2t}\right)$$

$$I=\sum_{k=0}^{\infty}\frac{(-1)^k}{k+\frac{1}{2}+2t}\ Rearrangement\ of\ series:$$

$$Bisection\ method$$

$$\sum_{n=0}^{\infty}(-1)^n f(n)=\sum_{k=0}^{\infty}\big(f(2k)-f(2k+1)\big)$$

$$I=\sum_{n=0}^{\infty}\frac{1}{2n+\frac{1}{2}+2t}-\sum_{n=0}^{\infty}\frac{1}{2n+\frac{3}{2}+2t}=\frac{1}{2}\sum_{n=0}^{\infty}\left(\frac{1}{n+\frac{1}{4}+t}-\frac{1}{n+\frac{3}{4}+t}\right)$$

$$Digamma\ function\ \ \psi(a)-\psi(b)=\sum_{n=0}^{\infty}\frac{1}{n+b}-\sum_{n=0}^{\infty}\frac{1}{n+a}$$

$$\int_0^{\infty}\frac{2\pi t}{ln^2(x)+4\pi^2t^2}\cdot\frac{dx}{1+x^2}=\frac{1}{2}\left(\psi\left(t+\frac{3}{4}\right)-\psi\left(t+\frac{1}{4}\right)\right)$$