ROMANIAN MATHEMATICAL MAGAZINE

Find:

$$I=\int_0^\infty \frac{1}{x}\ln(1+x)\ln\left(1+\frac{1}{x}\right)dx$$

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$$I=\int_0^\infty \frac{1}{x}\ln(1+x)\ln\left(1+\frac{1}{x}\right)dx=$$

$$=\int_0^1 \frac{1}{x}\ln(1+x)\ln\left(1+\frac{1}{x}\right)dx+\int_1^\infty \frac{1}{x}\ln(1+x)\ln\left(1+\frac{1}{x}\right)dx$$

$$I=I_1+I_2 \qquad I_2=\int_1^\infty \frac{1}{x}\ln(1+x)\ln\left(1+\frac{1}{x}\right)dx\to\left\{\frac{1}{x}=t\ \ dt=-t^2dx\ t\to 0,1\right\}$$

$$I_2=\int_0^1 \frac{\ln(1+t)\ln\left(1+\frac{1}{t}\right)}{t}dt=I_1\to \quad I=2I_1 \quad I=2\int_0^1\frac{\ln(1+t)}{t}\ln\left(1+\frac{1}{t}\right)dt$$

$$I=2\left(\int_0^1\frac{ln^2(1+t)}{t}dt-\int_0^1\frac{\ln(t)\ln(1+t)}{t}dt\right)=2(K-M)$$

$$K=\int_0^1\frac{ln^2(1+t)}{t}dt\to\frac{1}{1+t}=u\ \ t=\frac{1-u}{u}\quad du=-\frac{1}{(1+t)^2}dt\ \ du=-u^2dt$$

$$K=\int_{\frac{1}{2}}^1\frac{ln^2(u)}{u(1-u)}du=\int_{\frac{1}{2}}^1\frac{ln^2(u)}{u}du+\int_{\frac{1}{2}}^1\frac{ln^2(u)}{1-u}du$$

$$=\left[\frac{1}{3}ln^3(u)\right]_{\frac{1}{2}}^1+\int_0^1\frac{ln^2(u)}{1-u}du-\int_0^{\frac{1}{2}}\frac{ln^2(u)}{1-u}du=$$

$$=\frac{ln^3(2)}{3}+\sum_{n=0}^\infty\int_0^1u^n ln^2(u)du-\sum_{n=0}^\infty\int_0^{\frac{1}{2}}u^n ln^2(u)du=$$

$$=\frac{ln^3(2)}{3}+2\sum_{n=0}^\infty\frac{1}{(n+1)^3}-\sum_{n=0}^\infty\frac{ln^2(2)}{2^{n+1}n+1}+2\sum_{n=0}^\infty\frac{1}{n+1}\int_0^{\frac{1}{2}}u^n\ln(u)du=$$

$$=\frac{ln^3(2)}{3}+2\zeta(3)-ln^3(2)-2\sum_{n=0}^\infty\frac{\ln(2)}{2^{n+1}(n+1)^2}-2\sum_{n=0}^\infty\frac{1}{(n+1)^3 2^{n+1}}=$$

$$=\frac{-2ln^3(2)}{3}-2\zeta(3)-2\ln(2)\operatorname{Li}_2\left(\frac{1}{2}\right)-2\operatorname{Li}_3\left(\frac{1}{2}\right)=$$

$$= -\frac{2}{3}ln^3(2) + 2\zeta(3) - \left(-\frac{2}{3}ln^3(2) + \frac{7}{4}\zeta(3)\right) = \frac{1}{4}\zeta(3)$$

$$M = \int_0^1 \frac{\ln(x)\ln(1+x)}{x}dx = -\sum_{n=1}^{\infty}\frac{(-1)^n}{n}\int_0^1 x^{n-1}\ln(x)\,dx =$$

$$= \sum_{n=1}^{\infty}\frac{(-1)^n}{n^3} = -\eta(3) = -\frac{3}{4}\zeta(3)$$

$$\int_0^{\infty}\frac{1}{x}\ln(1+x)\ln\left(1+\frac{1}{x}\right)dx = 2(K-M) = 2\left(\frac{1}{4}\zeta(3) + \frac{3}{4}\zeta(3)\right) = 2\zeta(3)$$

$$Notes:\ Li_2\left(\frac{1}{2}\right) = \frac{\pi^2}{12} - \frac{ln^2(2)}{2}, \qquad Li_3\left(\frac{1}{2}\right) = \frac{7}{8}\zeta(3) + \frac{1}{6}ln^3(2) - \frac{\pi^2}{12}\ln(2)$$