ROMANIAN MATHEMATICAL MAGAZINE

Prove that:

$$\int_0^1 \frac{\ln(1-x^2)}{x^2}\arccos(x)\,dx = \frac{\pi^2}{4} - 4G$$

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$$I = \int_0^1 \frac{\ln(1-x^2)}{x^2}\arccos(x)\,dx \quad \arccos(x) = t \quad \frac{dt}{dx} = -\frac{1}{\sqrt{1-x^2}} = -\frac{1}{\sin(t)}$$

$$I = \int_0^{\frac{\pi}{2}} \frac{t\sin(t)\ln\left(\sin^2(t)\right)}{\cos^2(t)}dt = 2\int_0^{\frac{\pi}{2}} \frac{t\sin(t)\ln\left(\sin(t)\right)}{\cos^2(t)}dt$$

$$u = t\ln\left(\sin(t)\right), \qquad \frac{du}{dt} = \ln\left(\sin(t)\right) + t\cot(t)..$$

$\rightarrow IBP$

$$v = \int \frac{\sin(t)}{\cos^2(t)}dt \;\; \cos(t) = z \;\; dz = -\sin(t)dt \rightarrow v = -\int \frac{dz}{z^2} = \frac{1}{\cos(t)}$$

$$I = 2\left[\frac{t\ln\left(\sin(t)\right)}{\cos(t)}\right]_0^{\frac{\pi}{2}} - 2\int_0^{\frac{\pi}{2}} \frac{\ln\left(\sin(t)\right) + t\cot(t)}{\cos(t)}dt =$$

$$= -2\int_0^{\frac{\pi}{2}} \frac{\ln\left(\sin(t)\right)}{\cos(t)}dt - 2\int_0^{\frac{\pi}{2}} \frac{t}{\sin(t)}dt = -2I_1 - 2I_2$$

$$I_1 = \int_0^{\frac{\pi}{2}} \frac{\ln\left(\sin(t)\right)}{\cos(t)}dt \rightarrow \left\{\sin(t) = u \;\; \frac{du}{dt} = \cos(t)\right\}$$

$$I_1 = \int_0^1 \frac{\ln(u)}{1-u^2}du = \sum_{n=0}^{\infty}\int_0^1 u^{2n}\ln(u)\,du = -\sum_{n=0}^{\infty}\frac{1}{2n+1}\int_0^1 u^{2n}du =$$

$$= -\sum_{n=0}^{\infty}\frac{1}{(2n+1)^2} = -\zeta(2) + \frac{1}{2}\eta(2) = -\frac{\pi^2}{6} + \frac{\pi^2}{24} = -\frac{\pi^2}{8}$$

$$I_2 = \int_0^{\frac{\pi}{2}} \frac{t}{\sin(t)}dt \qquad \textit{Weierstrass Substitution}:$$

$$y = \tan\left(\frac{t}{2}\right) \;\; t = 2\arctan(y), \frac{dt}{dy} = \frac{2}{1+y^2}, \sin(t) = \frac{2\tan\left(\frac{t}{2}\right)}{1+\tan^2\left(\frac{t}{2}\right)} = \frac{2y}{1+y^2}$$

$$I_2=\int_0^1\frac{2\arctan(y)}{\left(\frac{2y}{1+y^2}\right)}\frac{2dy}{(1+y^2)}=2\int_0^1\frac{\arctan(y)}{y}dy=2\sum_{n=0}^{\infty}\frac{(-1)^n}{(2n+1)^2}=2G$$

$$I=\int_0^1\frac{\ln(1-x^2)}{x^2}\arccos(x)\,dx=-2I_1-2I_2=\frac{\pi^2}{4}-4G$$