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$$\text{Prove that: } \Omega = \int_0^{\frac{\pi}{2}}\int_0^{\frac{\pi}{2}}\int_0^{\frac{\pi}{2}} \frac{sen(2x)sen(2y)sen(2x)}{1+sen^2(x)+sen^2(x)sen^2(y)+sen^2(x)sen^2(y)sen^2(z)}dxdydz$$

$$= Li_3\left(-\frac{1}{2}\right)+Li_3\left(\frac{2}{3}\right)-Li_3\left(\frac{3}{4}\right)-Li_3\left(\frac{1}{3}\right)+2Li_2\left(-\frac{1}{2}\right)\ln(2)+\frac{13}{8}\zeta(3)+\frac{1}{6}\ln^3\left(\frac{3}{2}\right)$$

$$+\frac{1}{3}\ln^3(2)+\frac{1}{2}\ln^2\left(\frac{4}{3}\right)\ln(2)-\frac{1}{2}\ln^2\left(\frac{3}{2}\right)\ln(3)-\frac{\pi^2}{4}\ln(2)+\frac{\pi^2}{6}\ln(3)+\frac{1}{3}\ln^3(3)-\frac{1}{2}\ln(2)\ln^2(3)$$

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$$Let: \begin{cases} u = sen^2(x) \to du = sen(2x)dx \\ v = sen^2(y) \to dv = sen(2y)dy \Rightarrow \left[0,\frac{\pi}{2}\right] \to [0,1], \\ w = sen^2(z) \to dw = sen(2z)dz \end{cases} \quad \begin{matrix} u \to x \\ v \to y \\ w \to z \end{matrix}$$

$$\Omega = \int_0^1\int_0^1\int_0^1 \frac{1}{1+x+xy+xyz}dxdydz = \int_0^1\int_0^1 \frac{ln(1+x+2xy)-ln(1+x+xy)}{xy}dxdy =$$

$$= \int_0^1 \frac{1}{x}\left(\int_0^1 \frac{\ln(1+x+2xy)-\ln(1+x+xy)}{y}dy\right)dx$$

$$Let: J = \int_0^1 \frac{\ln(1+x+2xy)-\ln(1+x+xy)}{y}dy =$$

$$= \int_0^1 \frac{ln\left[(1+x)\left(1+\frac{2xy}{1+x}\right)\right]}{y}dy$$

$$-\int_0^1 \frac{ln\left[(1+x)\left(1+\frac{xy}{1+x}\right)\right]}{y}dy = \int_0^1 \frac{\ln\left(1+\frac{2xy}{1+x}\right)}{y}dy - \int_0^1 \frac{\ln\left(1+\frac{xy}{1+x}\right)}{y}dy$$

$$\#Note: \int_0^1 \frac{ln(1+ax)}{x}dx = -Li_2(-a) \ (\textit{Dilogarithm Function})$$

$$J = -Li_2\left(-\frac{2x}{1+x}\right)+Li_2\left(-\frac{x}{1+x}\right) \therefore$$

$$Then: \Omega = \int_0^1 \frac{1}{x}\left(-Li_2\left(-\frac{2x}{1+x}\right)+Li_2\left(-\frac{x}{1+x}\right)\right)dx$$

$$= \int_0^1 \frac{Li_2\left(-\frac{x}{1+x}\right)}{x}dx - \int_0^1 \frac{Li_2\left(-\frac{2x}{1+x}\right)}{x}dx = \Omega_1 - \Omega_2$$

$$\#\Omega_1 = \int_0^1 \frac{Li_2\left(-\frac{x}{1+x}\right)}{x}dx \ \to \ let: \begin{cases} t = \frac{x}{1+x} \to dt \to x = \frac{t}{1-t} \to \frac{dx}{x} = \left(\frac{1}{t}+\frac{1}{1-t}\right)dt \\ \qquad x = 0 \to t = 0 \wedge x = 1 \to t = \frac{1}{2} \end{cases}$$

$$\Omega_1=\int_0^{\frac{1}{2}} Li_2(-t)\left(\frac{1}{t}+\frac{1}{1-t}\right)dt=\int_0^{\frac{1}{2}}\frac{Li_2(-t)}{t}dt+\int_0^{\frac{1}{2}}\frac{Li_2(-t)}{1-t}dt$$

$$\#Note:\ \int_0^z\frac{Li_n(t)}{t}dt=Li_{n+1}(t)$$

$$\Omega_1=\mathrm{Li}_3\left(-\frac{1}{2}\right)+\int_0^{\frac{1}{2}}\frac{\mathrm{Li}_2(-\mathrm{t})}{1-\mathrm{t}}\mathrm{dt}=\begin{cases}u=Li_2(-t)\to du=-\dfrac{ln(1+t)}{t}\\ dv=\dfrac{dt}{1-t}\to v=-ln(1-t)\end{cases}\to\ \Omega_1=$$

$$=Li_3\left(-\frac{1}{2}\right)+\left[-Li_2(-t)\ln(1-t)\right]_0^{\frac{1}{2}}-\int_0^{\frac{1}{2}}\frac{\ln(1-t)\ln(1+t)}{t}dt$$

$$=Li_3\left(-\frac{1}{2}\right)+Li_2\left(-\frac{1}{2}\right)\ln(2)-\int_0^{\frac{1}{2}}\frac{\ln(1-t)\ln(1+t)}{t}dt$$

$$\#Note:\ln(1-x)\ln(1+x)=\frac{1}{2}\left[\ln^2(1-x^2)-\ln^2(1-x)-\ln^2(1+x)\right]$$

$$\Omega_1=Li_3\left(-\frac{1}{2}\right)+Li_2\left(-\frac{1}{2}\right)ln(2)-\frac{1}{2}\int_0^{\frac{1}{2}}\frac{1}{\mathrm{t}}\left[\ln^2(1-t^2)-\ln^2(1-t)-\ln^2(1+t)\right]dt$$

$$=Li_3\left(-\frac{1}{2}\right)+Li_2\left(-\frac{1}{2}\right)ln(2)$$

$$-\frac{1}{2}\int_0^{\frac{1}{2}}\frac{\ln^2(1-t^2)}{t}dt+\frac{1}{2}\int_0^{\frac{1}{2}}\frac{\ln^2(1-t)}{t}dt+\frac{1}{2}\int_0^{\frac{1}{2}}\frac{\ln^2(1+t)}{t}dt$$

$$p=t^2\to dp=2tdt\to\frac{dt}{t}=\frac{dp}{2p}$$

$$\Omega_1=Li_3\left(-\frac{1}{2}\right)+Li_2\left(-\frac{1}{2}\right)ln(2)$$

$$-\frac{1}{4}\int_0^{\frac{1}{4}}\frac{\ln^2(1-t)}{t}dt+\frac{1}{2}\int_0^{\frac{1}{2}}\frac{\ln^2(1-t)}{t}dt+\frac{1}{2}\int_0^{\frac{1}{2}}\frac{\ln^2(1+t)}{t}dt$$

$$\#Note:\ln^2(1-x)=2\sum_{k=1}^{\infty}\frac{H_{k-1}}{k}x^k\wedge ln^2(1+x)=2\sum_{k=1}^{\infty}\frac{(-1)^k}{k}H_{k-1}x^k$$

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$$\Omega_1 = Li_3\left(-\frac{1}{2}\right)+Li_2\left(-\frac{1}{2}\right)ln(2)$$

$$-\frac{1}{4}\int_0^{\frac{1}{4}}\frac{1}{t}\left(2\sum_{k=1}^{\infty}\frac{H_{k-1}}{k}t^k\right)dt+\frac{1}{2}\int_0^{\frac{1}{2}}\frac{1}{t}\left(2\sum_{k=1}^{\infty}\frac{H_{k-1}}{k}t^k\right)dt+\frac{1}{2}\int_0^{\frac{1}{2}}\frac{1}{t}\left(2\sum_{k=1}^{\infty}(-1)^k\frac{H_{k-1}}{k}t^k\right)dt=$$

$$=Li_3\left(-\frac{1}{2}\right)+Li_2\left(-\frac{1}{2}\right)ln(2)-\frac{1}{2}\sum_{k=1}^{\infty}\frac{H_{k-1}}{k}\int_0^{\frac{1}{4}}t^{k-1}dt+\sum_{k=1}^{\infty}\frac{H_{k-1}}{k}\int_0^{\frac{1}{2}}t^{k-1}dt+\sum_{k=1}^{\infty}(-1)^k\frac{H_{k-1}}{k}\int_0^{\frac{1}{2}}t^{k-1}dt$$

$$=Li_3\left(-\frac{1}{2}\right)+Li_2\left(-\frac{1}{2}\right)\ln(2)-\frac{1}{2}\sum_{k=1}^{\infty}\frac{H_k}{k^2}\left(\frac{1}{4}\right)^k+\frac{1}{2}\sum_{k=1}^{\infty}\frac{\left(\frac{1}{4}\right)^k}{k^3}+\sum_{k=1}^{\infty}\frac{H_k}{k^2}\left(\frac{1}{2}\right)^k-\sum_{k=1}^{\infty}\frac{\left(\frac{1}{2}\right)^k}{k^3}+\sum_{k=1}^{\infty}\frac{H_k}{k^2}\left(-\frac{1}{2}\right)^k$$

$$-\sum_{k=1}^{\infty}\frac{\left(-\frac{1}{2}\right)^k}{k^3}=Li_3\left(-\frac{1}{2}\right)+Li_2\left(-\frac{1}{2}\right)\ln(2)-\frac{1}{2}\sum_{k=1}^{\infty}\frac{H_{k-1}}{k^2}\left(\frac{1}{4}\right)^k+\sum_{k=1}^{\infty}\frac{H_{k-1}}{k^2}\left(\frac{1}{2}\right)^k+\sum_{k=1}^{\infty}\frac{(-1)^k}{k^2}H_{k-1}\left(\frac{1}{2}\right)^k$$

$$=Li_3\left(-\frac{1}{2}\right)+Li_2\left(-\frac{1}{2}\right)\ln(2)-\frac{1}{2}\sum_{k=1}^{\infty}\frac{H_k-\frac{1}{k}}{k^2}\left(\frac{1}{4}\right)^k+\sum_{k=1}^{\infty}\frac{H_k-\frac{1}{k}}{k^2}\left(\frac{1}{2}\right)^k+\sum_{k=1}^{\infty}\frac{(-1)^k}{k^2}\left(H_k-\frac{1}{k}\right)\left(\frac{1}{2}\right)^k$$

$$\#Note:\sum_{n=1}^{\infty}\frac{H_n}{n^2}x^n=$$

$$=Li_3(x)-Li_3(1-x)+\ln(1-x)\,Li_2(1-x)+\frac{1}{2}\ln(x)\ln^2(1-x)+\zeta(3)$$

$$Li_s(z)=\sum_{k=1}^{\infty}\frac{z^k}{k^s}\,(Polylogarithm\ function)$$

$$\Omega_1=Li_3\left(-\frac{1}{2}\right)+Li_2\left(-\frac{1}{2}\right)ln(2)+\frac{1}{2}Li_3\left(\frac{1}{4}\right)-Li_3\left(\frac{1}{2}\right)-Li_3\left(-\frac{1}{2}\right)$$

$$-\frac{1}{2}\left(Li_3\left(\frac{1}{4}\right)-Li_3\left(\frac{3}{4}\right)+ln\left(\frac{3}{4}\right)Li_2\left(\frac{3}{4}\right)+\frac{1}{2}ln\left(\frac{1}{4}\right)ln^2\left(\frac{3}{4}\right)+\zeta(3)\right)$$

$$+\left(Li_3\left(\frac{1}{2}\right)-Li_3\left(\frac{1}{2}\right)+\ln\left(\frac{1}{2}\right)Li_2\left(\frac{1}{2}\right)+\frac{1}{2}\ln\left(\frac{1}{2}\right)\ln^2\left(\frac{1}{2}\right)+\zeta(3)\right)$$

$$+\left(Li_3\left(-\frac{1}{2}\right)-Li_3\left(\frac{3}{2}\right)+ln\left(\frac{3}{2}\right)Li_2\left(\frac{3}{2}\right)+\frac{1}{2}ln\left(-\frac{1}{2}\right)ln^2\left(\frac{3}{2}\right)+\zeta(3)\right)$$

$$\#Note:\ \Re\left\{\ln\left(-\frac{1}{2}\right)\right\}=-\ln(2)\wedge\Re\left\{Li_2\left(\frac{3}{2}\right)\right\}=\frac{\pi^2}{3}-\frac{1}{2}\ln^2\left(\frac{3}{2}\right)-Li_2\left(\frac{2}{3}\right)$$

$$\Re\left\{Li_3\left(\frac{3}{2}\right)\right\}=Li_3\left(\frac{2}{3}\right)-\frac{1}{6}\ln^3\left(\frac{3}{2}\right)+\frac{\pi^2}{6}\ln\left(\frac{3}{2}\right)$$

$$\Omega_1=\frac{3}{2}\zeta(3)+2Li_3\left(-\frac{1}{2}\right)-Li_3\left(\frac{2}{3}\right)+\frac{1}{2}Li_3\left(\frac{3}{4}\right)-\frac{\pi^2}{6}\ln(2)-\ln\left(\frac{3}{2}\right)Li_2\left(\frac{2}{3}\right)-\frac{1}{2}\ln\left(\frac{3}{4}\right)Li_2\left(\frac{3}{4}\right)$$

$$+\frac{1}{2}\ln(2)\ln^2\left(\frac{3}{4}\right)-\frac{1}{3}\ln^3\left(\frac{3}{2}\right)\therefore$$

$$\#\Omega_2=\int_0^1\frac{Li_2\left(-\frac{2x}{1+x}\right)}{x}dx\rightarrow\ \ let:\begin{cases}t=\dfrac{2x}{1+x}\rightarrow dt\rightarrow x=\dfrac{t}{2-t}\rightarrow\dfrac{dx}{x}=\left(\dfrac{1}{t}+\dfrac{1}{2-t}\right)dt\\ x=0\rightarrow t=0\wedge x=1\rightarrow t=1\end{cases}$$

$$\Omega_2=\int_0^1 Li_2(-t)\left(\frac{1}{t}+\frac{1}{2-t}\right)dt=\int_0^1\frac{Li_2(-t)}{t}dt+\int_0^1\frac{Li_2(-t)}{2-t}dt$$

$$\Omega_2=\mathrm{Li}_3(-1)+\int_0^1\frac{\mathrm{Li}_2(-t)}{2-t}\mathrm{dt}=\begin{cases}u=Li_2(-t)\to\dfrac{du}{dt}=-\dfrac{ln(1+t)}{t}\\ dv=\dfrac{dt}{2-t}\to v=-ln(2-t)\end{cases}$$

$$\Omega_2=-\frac{3}{4}\zeta(3)+[-Li_2(-t)\ln(1-t)]_0^1-\int_0^1\frac{\ln(2-t)\ln(1+t)}{t}dt=-\frac{3}{4}\zeta(3)-\int_0^1\frac{\ln\left(2\left(1-\frac{t}{2}\right)\right)\ln(1+t)}{t}dt$$

$$=-\frac{3}{4}\zeta(3)$$
$$-\ln(2)\int_0^1\frac{\ln(1+t)}{t}dt$$
$$-\int_0^1\frac{\ln\left(1-\frac{t}{2}\right)\ln(1+t)}{t}dt=-\frac{3}{4}\zeta(3)-ln(2)\left(-Li_2(-1)\right)-\int_0^1\frac{ln\left(1-\frac{t}{2}\right)ln(1+t)}{t}dt$$

$$\Omega_2=-\frac{3}{4}\zeta(3)-\frac{\pi^2}{12}ln(2)-\int_0^1\frac{ln\left(1-\frac{t}{2}\right)\ln(1+t)}{t}dt\to\begin{cases}u=ln(1+t)\to du=\dfrac{dt}{1+t}\\ v=\displaystyle\int\frac{ln\left(1-\frac{t}{2}\right)}{t}dt\to v=-Li_2\left(\frac{t}{2}\right)\end{cases}$$

$$\Omega_2=-\frac{3}{4}\zeta(3)-\frac{\pi^2}{12}ln(2)-\left(-\frac{\pi^2}{12}\ln(2)+\frac{1}{2}\ln^3(2)+\int_0^1\frac{Li_2\left(\frac{t}{2}\right)}{1+t}dt\right)=$$

$$=-\frac{3}{4}\zeta(3)-\frac{1}{2}\ln^3(2)-\int_0^1\frac{Li_2\left(\frac{t}{2}\right)}{1+t}dt$$

$$=-\frac{3}{4}\zeta(3)-\frac{1}{2}ln^3(2)-\sum_{k=1}^{\infty}\frac{1}{2^kk^2}\int_0^1\frac{x^k}{1+x}dx$$

$$=-\frac{3}{4}\zeta(3)-\frac{1}{2}\ln^3(2)-\sum_{k=1}^{\infty}\frac{1}{2^kk^2}[(-1)^k(\ln(2)-\overline{H_k})$$

$$=-\frac{3}{4}\zeta(3)-\frac{1}{2}\ln^3(2)-\left(\ln(2)\sum_{k=1}^{\infty}\frac{\left(-\frac{1}{2}\right)^k}{k^2}-\sum_{k=1}^{\infty}\frac{\overline{H_k}}{k^2}\left(-\frac{1}{2}\right)^k\right)$$

$$=-\frac{3}{4}\zeta(3)-\frac{1}{2}\ln^3(2)-\left(\ln(2)\,Li_2\left(-\frac{1}{2}\right)-\sum_{k=1}^{\infty}\frac{\overline{H_k}}{k^2}\left(-\frac{1}{2}\right)^k\right)$$

$$\#Note: \sum_{k=1}^{\infty}\frac{\overline{H_k}}{k^2}x^k =$$

$$= Li_3\left(\frac{2x}{1+x}\right) - Li_3\left(\frac{x}{1+x}\right) - Li_3\left(\frac{1+x}{2}\right) - Li_3(-x) - Li_3(x) + Li_3\left(\frac{1}{2}\right) + \ln(1+x)\left[Li_2(x) + Li_2\left(\frac{1}{2}\right) + \frac{1}{2}\ln(2)\ln(1+x)\right]$$

$$\Omega_2 = -\frac{3}{4}\zeta(3) - \frac{1}{2}\ln^3(2) - \left(-\frac{\pi^2}{12}ln(2) - \frac{1}{6}ln^3(2) + 2ln(2)Li_2\left(-\frac{1}{2}\right) + 4Li_3\left(-\frac{1}{2}\right) + \frac{11}{4}\zeta(3)\right)$$

$$= \frac{\pi^2}{12}\ln(2) - \frac{1}{3}\ln^3(2) - 2\ln(2)Li_2\left(-\frac{1}{2}\right) - 4Li_3\left(-\frac{1}{2}\right) - \frac{7}{2}\zeta(3) \therefore$$

$Therefore$:

$$\Omega = \frac{3}{2}\zeta(3) + 2Li_3\left(-\frac{1}{2}\right) - Li_3\left(\frac{2}{3}\right) + \frac{1}{2}Li_3\left(\frac{3}{4}\right) - \frac{\pi^2}{6}\ln(2) - \ln\left(\frac{3}{2}\right)Li_2\left(\frac{2}{3}\right) - \frac{1}{2}\ln\left(\frac{3}{4}\right)Li_2\left(\frac{3}{4}\right) + \frac{1}{2}\ln(2)\ln^2\left(\frac{3}{4}\right) - \frac{1}{3}\ln^3\left(\frac{3}{2}\right) - \left(\frac{\pi^2}{12}\ln(2) - \frac{1}{3}\ln^3(2) - 2\ln(2)Li_2\left(-\frac{1}{2}\right) - 4Li_3\left(-\frac{1}{2}\right) - \frac{7}{2}\zeta(3)\right)$$

#Note: 1)Trilogarithm Inversion Formula (3/2 → 2/3):

$$\mathrm{Li}_3\left(\frac{3}{2}\right) = \mathrm{Li}_3\left(\frac{2}{3}\right) - \frac{1}{6}\ln^3\left(\frac{3}{2}\right) + \frac{\pi^2}{6}\ln\left(\frac{3}{2}\right)$$

2) Explicit Expansion of Constants:

$$Li_3\left(-\frac{1}{2}\right) = -\frac{5}{24}ln^3(2) + \frac{\pi^2}{24}ln(2) - \frac{3}{4}\zeta(3)$$

$$Li_2\left(-\frac{1}{2}\right) = \frac{1}{2}ln^2(2) - \frac{\pi^2}{12}$$

Applying these substitutions algebraically to our derived result yields the final compact form:

$$\Omega = Li_3\left(-\frac{1}{2}\right) + Li_3\left(\frac{2}{3}\right) - Li_3\left(\frac{3}{4}\right) - Li_3\left(\frac{1}{3}\right) + 2Li_2\left(-\frac{1}{2}\right)\ln(2) + \frac{13}{8}\zeta(3) + \frac{1}{6}\ln^3\left(\frac{3}{2}\right) + \frac{1}{3}\ln^3(2) + \frac{1}{2}\ln^2\left(\frac{4}{3}\right)\ln(2) - \frac{1}{2}\ln^2\left(\frac{3}{2}\right)\ln(3) - \frac{\pi^2}{4}\ln(2) + \frac{\pi^2}{6}\ln(3) + \frac{1}{3}\ln^3(3) - \frac{1}{2}\ln(2)\ln^2(3) \therefore$$