5801

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5801. Solve for real x:

$$\frac{1}{1+x^4}+\frac{1}{2+x^6}+\frac{1}{3+x^8}+\frac{1}{4+x^{10}}=\frac{77}{60x^2}$$

Daniel Sitaru

Solution 1 by Ángel Plaza, Universidad de Las Palmas de Gran Canaria, Spain.
It is enough to solve the equation for $x>0$. By changing variable by doing $\frac{1}{x^2}=t$, the equation changes to

$$\frac{t^2}{t^2+1}+\frac{t^3}{2t^3+1}+\frac{t^4}{3t^4+1}+\frac{t^5}{4t^5+1}=\frac{77t}{60}$$

and, since $t\neq 0$, it is equivalent to

$$\frac{t}{t^2+1}+\frac{t^2}{2t^3+1}+\frac{t^3}{3t^4+1}+\frac{t^4}{4t^5+1}=\frac{77}{60} \tag{8}$$

Let $f_p(t)=\frac{t^p}{pt^{p+1}+1}$, for $p=1,2,3,4$. $f_p'(t)=\frac{-pt^{p-1}(t^{p+1}-1)}{(pt^{p+1}+1)^2}$ whose only real root in the domain $t>0$ is $t=1$.

$$f_p''(t)=\frac{pt^{p-2}(-p(p+1)t^{p+1}+2pt^{2p+2}+p-1}{(pt^{p+1}+1)^3}$$

and

$$f_p''(1)=\frac{p(-p(p+5)+2p+p-1)}{(p+1)^3}=\frac{-p(p+1)^2}{(p+1)^3}<0.$$

Therefore, functions $f_p(t)$ have respective maximum at $t=1$, with value $f_p(1)=\frac{1}{p+1}$ and $\sum_{p=1}^4\frac{1}{p+1}=\frac{77}{60}$, which implies that the only solution to equation (8) is $t=1$, so the given equation has $x=1$ and $x=-1$ as the only real roots. □

Solution 2 by Brian D. Beasley, Simpsonville, SC..
Let $u=x^2$. Then we seek all non-negative real numbers u with

$$\frac{1}{1+u^2}+\frac{1}{2+u^3}+\frac{1}{2+u^4}+\frac{1}{4+u^5}=\frac{77}{60u}.$$

Noting that $u\neq 0$, we multiply by $f(u)=60u(1+u^2)(2+u^3)(3+u^4)(4+u^5)$ to obtain

$$\frac{f(u)}{1+u^2}+\frac{f(u)}{2+u^3}+\frac{f(u)}{3+u^4}+\frac{f(u)}{4+u^5}=77(1+u^2)(2+u^3)(3+u^4)(4+u^5).$$

Then tedious but straightforward algebra yields the equivalent equation $(u-1)^2g(u)=0$, where

$$g(u)=77u^{12}+94u^{11}+128u^{10}+256u^9+375u^8+716u^7+688u^6+$$
$$1070u^5+988u^4+1425u^3+1392u^2+696u+1848.$$

Since $g(u) > 0$ for all $u \geq 0$, we conclude that the unique non-negative real solution is $u = 1$. Hence the original equation has the real solutions $x = \pm 1$. □

Solution 3 by Daniel Văcaru, "Maria Teiuleanu" National Economic College, Piteşti, Romania.

By AM-GM, one has

$$x^4 + 1 \geq 2x^2 \Leftrightarrow \frac{1}{1+x^4} \leq \frac{1}{2x^2}$$

$$2 + x^6 = 1 + 1 + x^6 \geq 3\sqrt[3]{x^6} = 3x^2 \Leftrightarrow \frac{1}{2+x^6} \leq \frac{1}{3x^2}$$

$$3 + x^8 = 1 + 1 + 1 + x^8 \geq 4\sqrt[4]{x^8} = 4x^2 \Leftrightarrow \frac{1}{3+x^8} \leq \frac{1}{4x^2}$$

$$4 + x^{10} = 1 + 1 + 1 + 1 + x^{10} \geq 5\sqrt[5]{x^{10}} = 5x^2 \Leftrightarrow \frac{1}{4+x^{10}} \leq \frac{1}{5x^2}$$

It follows that

$$\frac{77}{60x^2} = \frac{1}{1+x^4} + \frac{1}{2+x^6} + \frac{1}{3+x^8} + \frac{1}{4+x^{10}} \leq \frac{1}{2x^2} + \frac{1}{3x^2} + \frac{1}{4x^2} + \frac{1}{5x^2} = \frac{77}{60x^2}.$$

One obtain

$$x^2 = 1$$

and one find

$$\mathcal{S} = \{-1, 1\}$$

where $\mathcal{S}$ denote the set of solutions of equation
$\frac{1}{1+x^4} + \frac{1}{2+x^6} + \frac{1}{3+x^8} + \frac{1}{4+x^{10}} = \frac{77}{60x^2}$. □

Solution 4 by David A. Huckaby, Angelo State University, San Angelo, TX.
For $x = 1$ or $x = -1$, the left side of the equation is $\frac{1}{2} + \frac{1}{3} + \frac{1}{4} + \frac{1}{5} = \frac{77}{60}$. So $x = 1$ and $x = -1$ are solutions of the equation. We will show that these are the only real solutions. Combining the fractions on the left side of the equation gives

$$\frac{x^{24} + x^{22} + x^{20} + 4x^{18} + 4x^{16} + 10x^{14} + 6x^{12} + 18x^{10} + 14x^8 + 19x^6 + 26x^4 + 50}{(x^4+1)(x^6+2)(x^8+3)(x^{10}+4)} = \frac{77}{6x^2},$$

so that

$$60x^2(x^{24} + x^{22} + x^{20} + 4x^{18} + 4x^{16} + 10x^{14} + 6x^{12} + 18x^{10} + 14x^8 + 19x^6 + 26x^4 + 50)$$
$$= 77(x^4+1)(x^6+2)(x^8+3)(x^{10}+4).$$

So

$$60x^2(x^{24} + x^{22} + x^{20} + 4x^{18} + 4x^{16} + 10x^{14} + 6x^{12} + 18x^{10} + 14x^8 + 19x^6 + 26x^4 + 50)$$
$$-77(x^4+1)(x^6+2)(x^8+3)(x^{10}+4) = 0,$$

that is,

$$77x^{28} - 60x^{26} + 17x^{24} + 94x^{22} - 9x^{20} + 222x^{18} - 369x^{16} + 410x^{14} - 464x^{12}$$
$$+546x^{10} - 524x^8 - 636x^6 + 1848x^4 - 3000x^2 + 1848 = 0.$$

Synthetic division reveals that $x = 1$ and $x = -1$ are actually double roots of this degree 28 polynomial, so we have

$$(x-1)^2(x+1)^2(77x^{24} + 94x^{22} + 128x^{20} + 256x^{18} + 375x^{16} + 716x^{14} + 688x^{12}$$
$$+1070x^{10} + 988x^8 + 1452x^6 + 1392x^4 + 696x^2 + 1848) = 0.$$

By Descartes' rule of signs, the degree 24 polynomial has no real positive or negative real roots. So $x = 1$ and $x = -1$ are indeed the only real roots of the original equation. $\square$

Solution 5 by Devis Alvarado, UNAH and UPNFM, Tegucigalpa, Hoduras.
Note that for every natural number $n \geq 2$, we have

$$n - 1 + x^{2n} = \underbrace{1 + 1 + \ldots + 1}_{n-1 veces} + x^{2n} \geq n\sqrt[n]{x^{2n}} = nx^2,$$

by the AM-GM inequality, which is equivalent to

$$\frac{1}{n - 1 + x^{2n}} \leq \frac{1}{nx^2}$$

Thus, we have

$$\sum_{k=2}^{n} \frac{1}{k - 1 + x^{2k}} \leq \sum_{k=2}^{n} \frac{1}{kx^2} = \left(\sum_{k=2}^{n} \frac{1}{k}\right)\frac{1}{x^2}.$$

This inequality becomes an equality when $x = \pm 1$. Thus, for the given problem, we use the previous inequality with $n = 5$, since $\frac{1}{2} + \frac{1}{3} + \frac{1}{4} + \frac{1}{5} = \frac{77}{60}$.

$$\frac{1}{1 + x^4} + \frac{1}{2 + x^6} + \frac{1}{3 + x^8} + \frac{1}{4 + x^{10}} \leq \frac{77}{60x^2}.$$

Therefore, the solutions are $x = \pm 1$. $\square$

Solution 6 by Michel Bataille, Rouen, France.
We remark that $\frac{77}{60} = \frac{1}{2} + \frac{1}{3} + \frac{1}{4} + \frac{1}{5}$, from which we deduce that 1 and -1 are solutions for x. We show that there are no other solutions.
Let x be a solution. Then the remark above yields

$$u_1(x) + u_2(x) + u_3(x) + u_4(x) = 0$$

where $u_n(x) = \frac{1}{n + x^{2n+2}} - \frac{1}{(n+1)x^2} = \frac{(n+1)x^2 - n - x^{2n+2}}{(n+1)x^2(n + x^{2n+2})}$.
Now, we have

$$\begin{aligned}(n+1)x^2 - n - x^{2n+2} &= n(x^2 - 1) + x^2(1 - (x^2)^n) \\ &= (1 - x^2)(x^2(1 + x^2 + \ldots + x^{2n-2}) - n) \\ &= (1 - x^2)((x^2 - 1) + (x^4 - 1) + \ldots + (x^{2n} - 1)) \\ &= (1 - x^2)(x^2 - 1)(1 + (x^2 + 1) + \ldots + (x^{2n-2} + \ldots + x^2 + 1)) \\ &= -(1 - x^2)^2 \cdot k(x)\end{aligned}$$

where $k(x) > 0$ for all real x. It follows that $u_n(x) \leq 0$ with equality if and only if $x = 1$ or $x = -1$. We see that the four terms in the left-hand side of (1) are less than or equal 0 so that (1) implies that each of them is 0, hence that $x = 1$ or $x = -1$. Thus, (1) implies that $x = 1$ or $x = -1$ and we are done. $\square$

Solution 7 by Moti Levy, Rehovot, Israel.
Set $t = x^2$, then the equation is equivalent to,

$$\frac{t}{1 + t^2} + \frac{t}{2 + t^3} + \frac{t}{3 + t^4} + \frac{t}{4 + t^5} = \frac{77}{60}.$$

Now we prove that

$$\frac{t}{1 + t^2} + \frac{t}{2 + t^3} + \frac{t}{3 + t^4} + \frac{t}{4 + t^5} \leq \frac{77}{60},$$

where equality is attained only or $t=1$. Define

$$F(t)=\frac{t}{1+t^2}+\frac{t}{2+t^3}+\frac{t}{3+t^4}+\frac{t}{4+t^5},\qquad t>0.$$

For $k=1,2,3,4$ let

$$f_k(t)=\frac{t}{k+t^{k+1}}.$$
$$f_k'(t)=\frac{k(1-t^{k+1})}{(k+t^{k+1})^2}.$$

For any fixed $k>0$, we have

$$\operatorname{sgn}\{f_k'(t)\}=\begin{cases}+ & (0<t<1),\\ 0 & (t=1),\\ - & (t>1).\end{cases}$$

Consequently,

$$F'(t)=\sum_{k=1}^{4}f_k'(t)\begin{cases}>0 & (0<t<1),\\ =0 & (t=1),\\ <0 & (t>1).\end{cases}$$

Thus F is strictly increasing on $(0,1)$ and strictly decreasing on $(1,\infty)$. Because of that monotonicity the *unique* global maximum of F on $(0,\infty)$ occurs at $t=1$.

$$F(1)=\frac{1}{1+1}+\frac{1}{2+1}+\frac{1}{3+1}+\frac{1}{4+1}=\frac{1}{2}+\frac{1}{3}+\frac{1}{4}+\frac{1}{5}=\frac{30+20+15+12}{60}=\frac{77}{60}$$

We conclude that for every $t>0$

$$F(t)\le F(1)=\frac{77}{60},$$

and the equality case of all four derivatives forces $t=1$, so equality is attained solely at that point.

Solution for x is $x=\pm1$. □

Solution 8 by Ulrich Abel, Technische Hochschule Mittelhessen, Friedberg, Germany.

We show the following generalization: The equation

$$\sum_{k=1}^{n}\frac{1}{k+x^{2k+2}}=\frac{H_{n+1}-1}{x^2}$$

where $H_n:=1+\frac{1}{2}+\ldots+\frac{1}{n}$ denotes the harmonic number, has exactly two real solutions, namely, $x=-1$ and $x=1$.

Problem #5801 SSMJ is the special case $n=4$ with $H_5-1=\frac{77}{60}$.

Proof: The substitution $t=x^2$ leads to the equation

$$f_n(t):=\sum_{k=1}^{n}\frac{t}{k+t^{k+1}}=H_{n+1}-1.$$

Since

$$f_n'(t)=\sum_{k=1}^{n}\frac{k(1-t^{k+1})}{(k+t^{k+1})^2}$$

we conclude that f_n is strictly increasing on $[0,1]$ and strictly decreasing on $[1,\infty)$. Hence, f unimodal function on $[0,\infty)$ with a unique maximum at $t=1$. Observing that $f_n(1) = H_{n+1} - 1$ we infer that $t = 1$ is the unique solution of the equation $f_n(t) = H_{n+1} - 1$. Therefore,

$$\frac{f_n(x^2)}{x^2} = \sum_{k=1}^{n} \frac{1}{k + x^{2k+2}} = \frac{H_{n+1} - 1}{x^2}$$

has exactly two solutions $x = -1$ and $x = 1$ on $\mathbb{R} \setminus \{0\}$. □

Solution 9 by proposer.

$$\frac{1}{x^2} + x^2 \overset{\text{AM-GM}}{\geq} 2 \cdot \sqrt{\frac{1}{x^2} \cdot x^2} = 2 \Rightarrow \frac{1}{\frac{1}{x^2} + x^2} \leq \frac{1}{2} \tag{1}$$

$$\frac{1}{x^2} + \frac{1}{x^2} + x^4 \overset{\text{AM-GM}}{\geq} 3 \cdot \sqrt[3]{\frac{1}{x^2} \cdot \frac{1}{x^2} \cdot x^4} = 3 \Rightarrow \frac{1}{\frac{2}{x^2} + x^4} \leq \frac{1}{3} \tag{2}$$

$$\frac{1}{x^2} + \frac{1}{x^2} + \frac{1}{x^2} + x^6 \overset{\text{AM-GM}}{\geq} 4 \cdot \sqrt[4]{\frac{1}{x^2} \cdot \frac{1}{x^2} \cdot \frac{1}{x^2} \cdot x^6} = 3 \Rightarrow \frac{1}{\frac{3}{x^2} + x^6} \leq \frac{1}{4} \tag{3}$$

$$\frac{1}{x^2} + \frac{1}{x^2} + \frac{1}{x^2} + \frac{1}{x^2} + x^8 \overset{\text{AM-GM}}{\geq} 5 \cdot \sqrt[5]{\frac{1}{x^2} \cdot \frac{1}{x^2} \cdot \frac{1}{x^2} \cdot \frac{1}{x^2} \cdot x^8} = 5$$

$$\Rightarrow \frac{1}{\frac{4}{x^2} + x^8} \leq \frac{1}{5} \tag{4}$$

By adding (1), (2), (3), (4):

$$\frac{1}{\frac{1}{x^2} + x^2} + \frac{1}{\frac{2}{x^2} + x^4} + \frac{1}{\frac{3}{x^2} + x^6} + \frac{1}{\frac{4}{x^2} + x^8} \leq \frac{1}{2} + \frac{1}{3} + \frac{1}{4} + \frac{1}{5}$$

$$\frac{x^2}{1 + x^4} + \frac{x^2}{2 + x^6} + \frac{x^2}{3 + x^8} + \frac{x^2}{4 + x^{10}} \leq \frac{30 + 20 + 15 + 12}{60}$$

$$\frac{1}{1 + x^4} + \frac{1}{2 + x^6} + \frac{1}{3 + x^8} + \frac{1}{4 + x^{10}} \leq \frac{77}{60x^2}$$

Equality holds for:

$$x^2 = \frac{1}{x^2}, x^4 = \frac{1}{x^2}, x^6 = \frac{1}{x^2}, x^8 = \frac{1}{x^2} \Rightarrow x = \pm 1$$

□

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