5793

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5793. Suppose $f : [a, b] \to [1, \infty)$ is a continuous function with $0 < a \le b$. Then:

$$n(b-a)^{n-1}\int_a^b f(x)dx \le (n-1)(b-a)^n + \left(\int_a^b f(x)dx\right)^n.$$

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Solution 1 by Ángel Plaza, Universidad de Las Palmas de Gran Canaria, Spain.
The proposed inequality may be written as

$$\frac{\int_a^b f(x)dx}{b-a} \le \frac{(n-1) + \left(\frac{\int_a^b f(x)dx}{b-a}\right)^n}{n},$$

which follows by the AM-GM inequality. □

Solution 2 by Brian Bradie, Department of Mahtematics. Christopher Newport University, Newport News, VA.
Let x and y be non-negative real numbers. By the arithmetic mean - geometric mean inequality,

$$(1) \qquad (n-1)x + y = \underbrace{x + x + \ldots + x}_{n-1 \text{ terms}} + y \ge n\sqrt[n]{x^{n-1}y}.$$

Because $f : [a, b] \to [1, \infty)$ is a continuous function and $a \le b$,

$$b - a \ge 0 \qquad \text{and} \qquad \int_a^b f(x)dx \ge 0.$$

Substituting

$$x = (b-a)^n \qquad \text{and} \qquad y = \left(\int_a^b f(x)dx\right)^n$$

into (1) then yields

$$(n-1)(b-a)^n + \left(\int_a^b f(x)dx\right)^n \ge n\sqrt[n]{(b-a)^{n(n-1)}\left(\int_a^b f(x)dx\right)^n}$$

$$= n(b-a)^{n-1}\int_a^b f(x)dx.$$

□

Solution 3 by David A. Huckaby, Angelo State University, San Angelo, TX.
If $a = b$, both sides of the inequality are zero, so we assume that $a < b$. Dividing both sides by $(b-a)^n$, the desired inequality is

$$n \cdot \frac{1}{b-a}\int_a^b f(x)dx \leq n - 1 + \left(\frac{1}{b-a}\int_a^b f(x)dx\right)^n.$$

We first note that this inequality does not holds for all real n. For example, letting $a = 0, b = 1, f(x) = 4$, and $n = \frac{1}{2}$, we have $\frac{1}{2}(4) \nleq \frac{1}{2} - 1 + (4)^{\frac{1}{2}}$. We will show that the inequality holds for $n \geq 1$ and for $n \leq 0$. That the inequality holds for $n = 0$ and for $n = 1$ can be seen be inspection, so we will consider the cases $n > 1$ and $n < 0$.
By the Mean Value Theorem for Integrals, there is a $c \in [a, b]$ such that $f(c) = \int_a^b f(x)dx$, so that the desired inequality is $nf(c) \leq n - 1 + [f(c)]^n$, that is, $n(f(c) - 1) \leq [f(c)]^n - 1$. If $f(c) = 1$, then both sides of this inequality are zero. Otherwise, $f(c) > 1$ by assumption, and we consider the equivalent inequality

$$n \leq \frac{[f(c)]^n - 1}{f(c) - 1}. \tag{2}$$

Let $g(x) = \frac{x^n - 1}{x-1}$ for $x > 1$. by l'Hôpital's Rule we have

$$\lim_{x\to 1^+} g(x) = \lim_{x\to 1^+} \frac{x^n - 1}{x - 1} = \lim_{x\to 1^+} \frac{nx^{n-1}}{1} = n.$$

So if we show that $g(x)$ is an increasing function for $x > 1$, then inequality (2) follows. To this end we note that $g'(x) = \frac{(n-1)x^{n+1} - nx^n + x}{(x-1)^2 x}$.
For the case $n > 1$, we rewrite $g'(x) = \frac{(n-1)x^n - nx^{n-1} + 1}{(x-1)^2}$. The denominator is clearly positive. Denoting the numerator by $h(x) = (n-1)x^n - nx^{n-1} + 1$, we have $h'(x) = n(n-1)x^{n-1} - n(n-1)x^{n-2} = n(n-1)(x^{n-1} - x^{n-2}) > 0$. Since $h(1) = 0$, this shows that $h(x) > 0$ for $x > 1$,
so that $g'(x) > 0$ for $x > 1$. Thus inequality (2) hols for $n \geq 1$.
(When $n > 1$ is an integer, a shorter route is to note that inequality (2) is simply $n \leq \frac{(f(c)-1)([f(c)]^{n-1} + [f(c)]}{f(c)-1}$ that is, $n \leq [f(c)]^{n-1} + [f(c))]^{n-2} + \ldots + f(c) + 1$, which is clearly true for $f(c) > 1$.)
For the case $n < 0$, we note that the denominator of $g'(x) = \frac{(n-1)x^{n+1} - nx^n + x}{(x-1)^2 x}$ is positive. Denoting the numerator by $k(x) = (n-1)x^{n+1} - nx^n + x$, we have
$k'(x) = (n+1)(n-1)x^n - n^2x^{n-1} + 1 = (n^2 - 1)x^n - n^2x^{n-1} + 1 = n^2(x^n - x^{n-1}) - x^n + 1 > 0$, since $x^n < 1$ for $n < 0$ and $x > 1$. Since $k(1) = 0$, this shows that $k(x) > 0$ for $x > 1$, so that $g'(x) > 0$ for $x > 1$. Thus inequality (2) also holds for $n < 0$.
So the original inequality holds for $n \geq 1$ and for $n \leq 0$.
Let $I = \int_a^b f(x)dx$. Since $b - a \geq 0$ and $I \geq 0$, we can apply the arithmetic mean - geometric mean inequality as follows:

$$(n-1)(b-a)^n + I^n = (b-a)^n + \ldots + (b-a)^n + I^n \geq n((b-a)^n \ldots (b-a)^n \cdot I^n)^{\frac{1}{n}}$$

and deduce that

$$(n-1)(b-a)^n + I^n \geq n((b-a)^{n(n-1)} \cdot I^n)^{\frac{1}{n}} = n(b-a)^{n-1}I,$$

as desired. □

Solution 5 by Perfetti Paolo, dipartimento di matematica Universita di. "Tor Vergata", Roma, Italy.

$$\left(\int_a^b f(x)dx\right)^n + \underbrace{(b-a)^n + \ldots (b-a)^n}_{n-1\ times} \geq \left(\left(\int_a^b f(x)dx\right)^n (b-a)^{n(n-1)}\right)^{\frac{1}{n}} =$$

$$= n(b-a)^{n-1} \int_a^b f(x)dx$$

□

Solution by Ulrich Abel, Technische Hochschule Mittelhessen, Friedberg, Germany.

As $f(x) \geq 1$, we have $\int_a^b f(x)dx = (b-a)(1+y)$ with a real number $y \geq 0$. We have to show that

$$n(b-a)^n(1+y) \leq (n-1)(b-a)^n + (b-a)^n(1+y)^n.$$

If $a = b$ the inequality is obvious. In case $a < b$ the inequality is equivalent to

$$n(1+y) \leq n - 1 + (1+y)^n$$

or, after an elementary simplification, to

$$1 + ny \leq (1+y)^n.$$

This is just the Bernoulli inequality. □

Solution 7 by Albert Stadler, Herrliberg, Switzerland.
We assume that n is a real variable with $n \geq 1$. The inequality holds trivially true if $b = a$. Let $b > a$, and let $I := \frac{1}{b-a}\int_a^b f(x)dx$. The inequality then reads as

$$nI \leq n - 1 + I^n$$

which is exactly Bernoulli's inequality
(see for instance https://en.wikipedia.org/wiki/Bernoulli%27s_inequality):

$$(1+x)^r \geq 1 + rx$$

for every real number $r \geq 1$ and $x \geq -1$. The inequality is strict if $x \neq 0$ and $r \neq 1$. Hence the asumption that $f(x) \geq 1$ is not requiered. It suffices to assume that f is nonnegative. □

Solution 8 by proposer.
We prove by induction that:

$$x_1, x_2, \ldots, x_n \in [1, \infty); n \in \mathbb{N}^* \text{ implies:}$$

$$x_1 + x_2 + \ldots + x_n \leq n - 1 + x_1 x_2 \ldots x_n$$

$$\text{Checking: } n = 1; x_1 = 1 - 1 + x_1 \Leftrightarrow x_1 \leq x_1$$

$$n = 2 : x_1 + x_2 \leq 1 + x_1 x_2 \Leftrightarrow (x_1 - 1)(x_2 - 1) \geq 0. \text{ True.}$$

$$P(k) : x_1 + x_2 + \ldots + x_k \leq k - 1 + x_1 x_2 \ldots x_k \tag{1}$$

Suppose that it's true.

$$P(k+1) : x_1 + x_2 + \ldots + x_k + x_{k+1} \leq k + x_1 x_2 \ldots x_k x_{k+1} \text{ (to prove)}$$

$$x_1 + x_2 + \ldots + x_k + x_{k+1} \overset{P(k)}{\leq} k - 1 + x_1 x_2 \ldots x_k + x_{k+1}$$

Remains to prove that:

$$k-1+x_1x_2\dots x_k+x_{k+1}\le k+x_1x_2\dots x_kx_{k+1}$$
$$x_1x_2\dots x_kx_{k+1}-x_1x_2\dots x_k-x_{k+1}+1\ge 0$$
$$x_1x_2\dots x_k(x_{k+1}-1)-(x_{k+1}-1)\ge 0$$
$$(x_{k+1}-1)(x_1x_2\dots x_k-1)\ge 0 \text{ which is true because}$$
$$x_{k+1}\ge 1; x_1x_2\dots x_k\ge 1$$
$$P(k)\to P(k+1)$$

In (1) we take:

$$x_1\to f(x_1); x_2\to f(x_2);\dots; x_n\to f(x_n)$$

$$\sum_{k=1}^{n} f(x_k)\le n-1+\prod_{k=1}^{n} f(x_k) \tag{2}$$

By integration in (2):

$$\int_a^b\int_a^b\cdot\ldots\cdot\int_a^b\left(\sum_{k=1}^{n} f(x_k)\right)dx_1dx_2\cdot\ldots\cdot dx_n\le$$

$$\le\int_a^b\int_a^b\cdot\ldots\cdot\int_a^b(n-1)dx_1dx_2\dots dx_n+\int_a^b\int_a^b\cdot\ldots\cdot\int_a^b\prod_{k=1}^{n} f(x_k)dx_1dx_2\dots dx_n$$

$$\sum_{k=1}^{n}\int_a^b\int_a^b\cdot\ldots\cdot\int_a^b f(x_k)dx_1dx_2\dots dx_n\le$$

$$\le(n-1)(b-a)^n+\prod_{k=1}^{n}\int_a^b f(x_k)dx_k$$

$$n(b-a)^{n-1}\int_a^b f(x)dx\le(n-1)(b-a)^n+\left(\int_a^b f(x_k)dx_k\right)^n$$

Equality holds for $a=b$ or $f(x)\equiv 1$. □

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