5789

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5789. Let $0 < a \leq b$. Suppose $f : [a, b] \to (0, \infty)$ is a continuous function. Then:

$$\int_a^b \int_a^b \int_a^b \Big(f^2(x)+f^2(y)+f^2(z)\Big)^2 dxdydz \geq 9(b-a)\left(\int_a^b f(x)dx\right)\left(\int_a^b f^3(x)dx\right).$$

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Solution 1 by Michel Bataille, Rouen, France.
We use the following lemma: if a, b, c are real numbers, then
$(a^2 + b^2 + c^2)^2 \geq 3(a^3b + b^3c + c^3a)$.
Proof (from Z. Cvetkovski, *Inequalities*, Springer, 2012, p. 227)
Let $x = a^2 - ab + bc, y = b^2 - bc + ca, z = c^2 - ca + ab$. The inequality directly follows from the well-known $(x + y + z)^2 \geq 3(xy + yz + zx)$.
From this lemma, we have

$$(f^2(x) + f^2(y) + f^2(z))^2 \geq 3(f^3(x)f(y) + f^3(y)f(z) + f^3(z)f(x))$$

for all $(x, y, z) \in [a, b]^3$. Integrating, we obtain

$$\int_a^b \int_a^b \int_a^b (f^2(x) + f^2(y) + f^2(z))^2 dxdydz \geq$$

$$3\left((b-a)\int_a^b \int_a^b f^3(x)f(y)dxdy+(b-a)\int_a^b \int_a^b f^3(y)f(z)dydz+(b-a)\int_a^b \int_a^b f^3(z)f(x)dxdz\right)$$

Now, if $I = \int_a^b f^3(x)$ and $J = \int_a^b f(x)dx$, we have

$$\int_a^b \int_a^b f^3(x)f(y)dxdy = \left(\int_a^b f^3(x)dx\right)\left(\int_a^b f(y)dy\right) = I \cdot J$$

and similarly,

$$\int_a^b \int_a^b f^3(y)f(z)dydz = I \cdot J = \int_a^b \int_a^b f^3(z)f(x)dxdz$$

Thus, we have

$$\int_a^b \int_a^b \int_a^b (f^2(x) + f^2(y) + f^2(z))^2 dxdydz \geq 3(3(b-a)I \cdot J) = 9(b-a)IJ,$$

as desired. □

Solution 2 by Perfetti Paolo, dipartimento di matematica. Universita di "Tor Vergata", Roma, Italy.
The inequality is

$$\int_a^b \int_a^b \int_a^b (f^4(x) + f^4(y) + f^4(z) + 2(f(x)f(y))^2+$$

$$+2(f(y)f(z))^2+2(f(x)f(z))^2)dxdydz+$$

$$\geq 9(b-a)\left(\int_a^b f(x)dx\right)\left(\int_a^b f^3(x)dx\right)$$

or

$$3(b-a)^2\int_a^b f^4(x)dx+6(b-a)\left(\int_a^b f^2(x)dx\right)\left(\int_a^b f^2(y)dy\right)$$

$$\geq 9(b-a)\left(\int_a^b f(x)dx\right)\left(\int_a^b f^3(x)dx\right)$$

$$(b-a)\int_a^b f^4(x)dx+2\left(\int_a^b f^2(x)dx\right)\left(\int_a^b f^2(y)dy\right)\geq$$

$$\geq 3\left(\int_a^b f(x)dx\right)\left(\int_a^b f^3(x)dx\right)$$

This may be rewritten as

$$\int_a^b\int_a^b f^4(x)dxdy+2\int_a^b\int_a^b f^2(x)f^2(y)dxdy\geq 3\int_a^b\int_a^b f(x)f^3(y)dxdy$$

that is

$$\int_a^b\int_a^b (f^4(x)+2f^2(x)f^2(y)-3f(x)f^3(y))dxdy$$

or

$$\int_a^b\int_a^b f(x)(f(x)-f(y))(f^2(x)+f(x)f(y)+3f^2(y))dxdy$$

Now let's consider the two points (x,y) and (y,x). The sum of the two terms are

$$f(x)f(x)-f(y))(f^2(x)+f(x)f(y)+3f^2(y))+$$

$$+f(y)(f(y)-f(x))(f^2(y)+f(y)f(x)+3f^2(x))=$$

$$=(f(x)-f(y))^2(f^2(x)+f^2(y)+3f(x)f(y))\geq 0$$

thus concluding the proof. □

Solution 3 by proposer.
Lemma: If $a,b,c\in\mathbb{R}$ then:

$$(a^2+b^2+c^2)^2\geq 3(a^3b+b^3c+c^3a)$$

Proof.

$$0\leq\sum_{cyc}(a^2-b^2-ab-ac+2bc)^2=$$

$$=\sum_{cyc}a^4+\sum_{cyc}b^4+\sum_{cyc}a^2b^2+\sum_{cyc}a^2c^2+4\sum_{cyc}b^2c^2-$$

$$-2\sum_{cyc}a^2b^2-2\sum_{cyc}a^3b-2\sum_{cyc}a^3c+4abc\sum_{cyc}a+$$

$$+2\sum_{cyc}ab^3+2abc\sum_{cyc}b-4\sum_{cyc}b^3c+2abc\sum_{cyc}c-$$

$$-4abc\sum_{cyc}b-4abc\sum_{cyc}c=$$

$$= 2\sum_{cyc} a^4 + 4\sum_{cyc} b^2c^2 - 6\sum_{cyc} a^3b =$$
$$= 2(a^2+b^2+c^2)^2 - 6(a^3b+b^3c+c^3a)$$
$$0 \le 2(a^2+b^2+c^2)^2 - 6(a^3b+b^3c+c^3a)$$
$$(a^2+b^2+c^2)^2 \ge 3(a^3b+b^3c+c^3a) \tag{1}$$

Let's take in $(1) : a = f(x); b = f(y); c = f(z)$

$$(f^2(x)+f^2(y)+f^2(z))^2 \ge 3(f^3(x)f(y)+f^3(y)f(z)+f^3(z)f(x))$$
$$\int_a^b\int_a^b\int_a^b (f^2(x)+f^2(y)+f^2(z))^2 dxdydz \ge$$
$$\ge 3\sum_{cyc}\int_a^b\int_a^b\int_a^b f^3(x)f(y)dxdydz =$$
$$= 3\sum_{cyc}\left(\int_a^b f^3(x)dx\right)\left(\int_a^b f(y)dy\right)\left(\int_a^b dz\right) =$$
$$= 3(b-a)\sum_{cyc}\left(\int_a^b f^3(x)dx\right)\left(\int_a^b f(x)dx\right) =$$
$$= 9(b-a)\left(\int_a^b f(x)dx\right)\left(\int_a^b f^3(x)dx\right)$$

Equality holds for $a = b$ or $f \equiv 1$. □

□

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