5781

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5781. Let $m, n, p, q, r, s \in \mathbb{N} \setminus \{0\}$ and define

$$H_n^{(m)} = \frac{1}{1^m} + \frac{1}{2^m} + \ldots + \frac{1}{n^m}.$$

Prove that

$$(H_n^{(2p)} + H_n^{(2q)})(H_n^{(2r)} + H_n^{(2s)}) \geq (H_n^{(p+r)} + H_n^{(q+s)})^2.$$

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Solution 1 by Michel Bataille, Rouen, France.
Let $a_j = \frac{1}{j^p}$ for $j = 1, 2, \ldots, n$ and $a_j = \frac{1}{(j-n)^q}$ for $j = n+1, n+2, \ldots, 2n$. Similarly, let $b_j = \frac{1}{j^r}$ for $j = 1, 2, \ldots, n$ and $b_j = \frac{1}{(j-n)^s}$ for $j = n+1, n+2, \ldots, 2n$. Then the Cauchy - Schwarz inequality gives

$$\left(\sum_{j=1}^{2n} a_j^2\right)\left(\sum_{j=1}^{2n} b_j^2\right) \geq \left(\sum_{j=1}^{2n} a_j b_J\right)^2,$$

which is nothing else than

$$(H_n^{(2p)} + H_n^{(2q)})(H_n^{(2r)} + H_n^{(2s)}) \geq (H_n^{(p+r)} + H_n^{(q+s)})^2.$$

□

Solution 2 by Perfetti Paolo, dipartimento de matematica.
Universita di "Tor Vergata", Roma, Italy.
Cauchy - Schwarz yields

$$(H_n^{(2p)} + H_n^{(2q)})(H_n^{(2r)} + H_n^{(2s)}) \geq \left(\sqrt{H_n^{(2p)} H_n^{(2r)}} + \sqrt{H_n^{(2q)} H_n^{(2s)}}\right)^2$$

hence we come to

$$\sqrt{H_n^{(2p)} H_n^{(2r)}} + \sqrt{H_n^{(2q)} H_n^{(2s)}} \geq H_n^{(p+r)} + H_n^{(q+s)} \tag{1}$$

By Cauchy - Schwarz again

$$H_n^{(2p)} H_n^{(2r)} = \sum_{k=1}^{n} \frac{1}{k^{2p}} \sum_{k=1}^{n} \frac{1}{k^{2r}} \geq \left(\sum_{k=1}^{n} \frac{1}{k^p} \frac{1}{k^r}\right)^2 = \left(\sum_{k=1}^{n} \frac{1}{k^{r+p}}\right)^2 = (H_n^{(r+p)})^2$$

$$H_n^{(2q)} H_n^{(2s)} = \sum_{k=1}^{n} \frac{1}{k^{2q}} \sum_{k=1}^{n} \frac{1}{k^{2s}} \geq \left(\sum_{k=1}^{n} \frac{1}{k^q} \frac{1}{k^s}\right)^2 = \left(\sum_{k=1}^{n} \frac{1}{k^{q+s}}\right)^2 = (H_n^{(q+s)})^2$$

and (1) clearly follows. □

Solution 3 by proposer.
By Huygens inequality:

$$\left(H_n^{(2p)}+H_n^{(2q)}\right)\left(H_n^{(2r)}+H_n^{(2s)}\right)\geq$$
$$\geq\left(\sqrt{H_n^{(2p)}\cdot H_n^{(2r)}}+\sqrt{H_n^{(2q)}\cdot H_n^{(2s)}}\right)^2=$$
$$=\left(\sqrt{\left(\sum_{k=1}^{n}\frac{1}{k^{2p}}\right)\left(\sum_{k=1}^{n}\frac{1}{k^{2r}}\right)}+\sqrt{\left(\sum_{k=1}^{n}\frac{1}{k^{2q}}\right)\left(\sum_{k=1}^{n}\frac{1}{k^{2s}}\right)}\right)^2\geq$$
$$\overset{\text{Cauchy-Schwarz}}{\geq}\left(\sqrt{\left(\sum_{k=1}^{n}\frac{1}{k^{p}}\cdot\frac{1}{k^{r}}\right)^2}+\sqrt{\left(\sum_{k=1}^{n}\frac{1}{k^{q}}\cdot\frac{1}{k^{s}}\right)^2}\right)^2=$$
$$=\left(\sum_{k=1}^{n}\frac{1}{k^{p+r}}+\sum_{k=1}^{n}\frac{1}{k^{q+s}}\right)^2=$$
$$=\left(H_n^{(p+r)}+H_n^{(q+s)}\right)^2$$

□

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