# 5779

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If $0 < a \leq b$ then:

$$e^{ab} + e^{(\frac{2ab}{a+b})^2} \leq e^{(\frac{2ab}{a+b})^2} + e^{(\sqrt{ab}+\frac{a+b}{2}-\frac{2ab}{a+b})^2}$$

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*Solution 1 by Albert Stadler, Herrliberg, Switzerland.*
We divide both sides by $e^{(\frac{2ab}{a+b})^2} > 0$ and get the equivalent inequality

$$e^{\frac{ab(a-b)^2}{(a+b)^2}} + e^{\frac{(a-b)^2(a^2+6ab+b^2)}{4(a+b)^2}} \leq 1 + e^{\frac{(a-b)^2(a+4\sqrt{ab}+b)}{4(a+b)}}.$$

Let $x$ be real. The Taylor expansion of the exponential function is

$$e^x = \sum_{k=0}^{\infty} \frac{x^k}{k!.}$$

It is therefore sufficient to prove that

$$\Big(\frac{ab(a-b)^2}{(a+b)^2}\Big)^k + \Big(\frac{(a-b)^2(a^2+6ab+b^2)}{4(a+b)^2}\Big)^k \leq \Big(\frac{(a-b)^2(a+4\sqrt{ab}+b)}{4(a+b)}\Big)^k$$

for all $k \geq 1$. This inequality is equivalent to

$$(4ab)^k + (a^2+6ab+b^2)^k \leq \Big((a+4\sqrt{ab}+b)(a+b)\Big)^k, \quad k = 1, 2, 3, \ldots$$

which is true, since

$$(a+4\sqrt{ab}+b)(a+b) = (a+b)^2+4\sqrt{ab}(a+b) \geq (a+b)^2+4\sqrt{ab}\cdot 2\sqrt{ab} = a^2+10ab+b^2$$

and given $x, y > 0$ we have $x^k + y^k \leq (x+y)^k$ for $k = 1, 2, 3, \ldots$ $\square$

*Solution 2 by Michel Bataille, Rouen, France.*
If $a = b$ equality holds, so we suppose that $a < b$ in what follows. Let $h = 2ab/(a+b), g = \sqrt{ab}, m = (a+b)/2$. We have $h < g < m$ (inequalities of means). The proposed inequality writes as

$$\phi(g) - \phi(h) \leq \phi(g+m-h) - \phi(m) \tag{1}$$

where $\phi(x) = e^{x^2}$. Note that $h < g < m < g+m-h$. The Mean Value Theorem show that $\phi(g)-\phi(h) = (g-h)\phi'(u)$ and $\phi(g+m-h)-\phi(m) = (g-h)\phi'(v)$ for some $u \in (h, g), v \in (m, g+m-h)$. An easy calculation gives $\phi''(x) = (4x^2+2)e^{x^2} > 0$, hence $\phi'$ is an increasing function. Consequently $\phi'(u) < \phi'(v)$ and since $g-h > 0$, (1) follows. $\square$

*Solution 3 by Perfetti Paolo, dipatimento di matematica.*
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Set

$$\sqrt{ab} =: G, \quad \frac{a+b}{2} =: M, \quad \frac{2ab}{a+b} =: H.$$

We know $M \geq G \geq H$. This inequality in question is equivalent to
$$e^{G^2} - e^{H^2} \leq e^{(G+M-H)^2} - e^{M^2}.$$
We observe that $(G+M-H)^2 - M^2 = (G-H)(G+2M-H) \geq 0$. Clearly
$$H^2 \leq G^2 \leq M^2 \leq (G+M-H)^2$$
Lagrange's theorem yields
$$e^{G^2} - e^{H^2} - e^{\xi}(G^2 - H^2) \leq e^{(G+M-H)^2} - e^{M^2} = e^{\eta}\Big((G+M-H)^2 - M^2\Big)$$
where $G^2 < \xi < H^2 \leq M^2 < \eta < (G+M-H)^2$. Because of $(e^x)^\eta = e^x > 0$ we have
$$e^{\eta}\Big((G+M-H)^2 - M^2\Big) \geq e^{\xi}\Big((G+M-H)^2 - M^2\Big)$$
thus it suffices to show that
$$e^{\xi}(G^2 - H^2) \leq e^{\xi}\Big((G+M-H)^2 - M^2\Big) \Leftrightarrow G^2 - H^2 \leq (G+M-H)^2 - M^2$$
or
$$GH + MH \leq H^2 + GM \Leftrightarrow H(M-H) \leq G(M-H)$$
and this concludes the proof. □

*Solution 4 by proposer.*
Lemma:
If $a, b, x, y, z \in \mathbb{R}; a \leq x \leq y \leq z \leq b; y + z \leq b + x; f : [a, b] \to \mathbb{R}; f$ convexe; then:
$$f(y) + f(z) \leq f(x) + f(y+z-x) \tag{1}$$

*Proof.*
$$x \leq z \Rightarrow 0 \leq z - x \Rightarrow y \leq y + z - x$$
$$x \leq y \Rightarrow 0 \leq y - x \Rightarrow z \leq y + z - x$$
$$y \in [x; y+z-x] \Rightarrow (\exists)\alpha \in [0, 1]$$
$$y = \alpha x + (1-\alpha)(y+z-x) \tag{2}$$
$$z \in [x; y+z-x] \Rightarrow (\exists)\beta \in [0, 1]$$
$$z = \beta x + (1-\beta)(y+z-x) \tag{3}$$
By adding $(2); (3)$:
$$y + z = (\alpha+\beta)x + (y+z-x)(2-\alpha-\beta) \tag{4}$$
$$y + z = x + (y+z-x)$$
Replacing $y+z$ in (4):
$$x + (y+z-x) = (\alpha+\beta)x + (y+z-x)(2-\alpha-\beta)$$
$$(\alpha+\beta-1)x + (y+z-x)(1-\alpha-\beta) = 0$$
$$(\alpha+\beta-1)(x-y-z+x) = 0$$
$$(\alpha+\beta-1)(2x-y-z) = 0$$
$$\text{Case I: } 2x - y - z = 0 \Rightarrow y + z = 2x \tag{5}$$
$$\text{But } x \leq y; x \leq z \Rightarrow 2x \leq y + z \tag{6}$$

By $(5);(6) \Rightarrow x=y=z$
Inequality (1) becomes:

$$f(x)+f(x) \leq f(x)+f(x+x-x) \text{ (true)}$$

$$\text{Case II: } \alpha+\beta-1=0 \Rightarrow \alpha+\beta=1 \Rightarrow 2-\alpha-\beta=1$$

$$f \text{ convexe }; y,z \in [x, y+z-x] \Rightarrow$$

$$\Rightarrow (\exists)\alpha,\beta \in [0,1] \text{ such that:}$$

$$f(y) \leq \alpha f(x)+(1-\alpha)f(y+z-x) \tag{7}$$

$$f(z) \leq \beta f(x)+(1-\beta)f(y+z-x) \tag{8}$$

By adding $(7);(8)$:

$$f(y)+f(z) \leq (\alpha+\beta)f(x)+(2-\alpha-\beta)f(y+z-x)$$

$$f(y)+f(z) \leq f(x)+f(y+z-x)$$

□

Back to main problem:
We take in $(1): f:(0,\infty) \to \mathbb{R}$;

$$f(x)=e^{x^2}; f'(x)=2xe^{x^2};$$

$$f''(x)=2e^{x^2}(1+2x^2)>0; f \text{ convexe.}$$

$$f(y)+f(z) \leq f(x)+f(y+z-x)$$

$$e^{y^2}+e^{z^2} \leq e^{x^2}+e^{(y+z-x)^2} \tag{9}$$

We take in (9):

$$x=\frac{2ab}{a+b}; y=\sqrt{ab}; z=\frac{a+b}{2}$$

$$e^{(\sqrt{ab})^2}+e^{(\frac{a+b}{2})^2} \leq e^{(\frac{2ab}{a+b})^2}+e^{(\sqrt{ab}+\frac{a+b}{2}-\frac{2ab}{a+b})^2}$$

$$e^{ab}+e^{(\frac{a+b}{2})^2} \leq e^{(\frac{2ab}{a+b})^2}+e^{(\sqrt{ab}+\frac{a+b}{2}-\frac{2ab}{a+b})^2}$$

Equality holds for $a=b$. □

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