5570

DANIEL SITARU - ROMANIA

5570. Suppose $a, b \in \mathbb{C}$ with $|a^2+1| \leq 1, |a^4+1| \leq 1, |b^3+1| \leq 1, |b^6+1| \leq 1$. Prove that:

$$|a+b|^2 + |a-b|^2 \leq 4$$

Proposed by Daniel Sitaru - Romania

Solution 1 by Michel Bataille, Rouen, France.
We have $|a+b|^2 = (a+b)(\overline{a}+\overline{b}) = |a|^2 + a\overline{b} + \overline{a}b + |b|^2$ and $|a-b|^2 = |a|^2 - a\overline{b} - \overline{a}b + |b|^2$, hence $|a+b|^2 + |a-b|^2 = 2(|a|^2 + |b|^2)$. Thus, it suffices to show that $|a|^2 + |b|^2 \leq 2$ or even that $|a| \leq 1$ and $|b| \leq 1$.
If $2|a^2| \leq 1$, then certainly $|a| \leq 1$. If $2|a|^2 \geq 1$, then using the triangular inequality, we see that

$$1 \geq |a^4+1| = |(a^2+1)^2 - 2a^2| \geq \left| |a^2+1|^2 - 2|a|^2 \right| = 2|a|^2 - |a^2+1|^2$$

so that $2 \geq 1 + |a^2+1|^2 \geq 2|a^2| = 2|a|^2$ and $|a| \leq 1$ follows.
Similarly, we have $|b| \leq 1$ if $2|b|^3 = |2b^3| \leq 1$. If $|2b^3| \geq 1$, then, as above,

$$1 \geq |b^6+1| = |(b^3+1)^2 - 2b^3| \geq \left| |b^3+1|^2 - |2b^3| \right| = |2b^3| - |b^3+1|^2$$

hence $2 \geq 1 + |b^3+1|^2 \geq 2|b|^3$ and $|b| \leq 1$ follows.
In any case, we have $|a| \leq 1$ and $|b| \leq 1$. □

Solution 2 by proposer.

$$2|a^4| = |2a^4| = |(a^2+1)^2 - (a^4+1)| \leq$$
$$\leq |(a^2+1)^2| + |a^4+1| = |a^2+1|^2 + |a^4+1| \leq 1+1 = 2$$
$$\Rightarrow 2|a^4| \leq 2 \Rightarrow |a^4| \leq 1 \Rightarrow (|a|)^4 \leq 1 \Rightarrow |a| \leq 1$$
$$2|b^6| = |2b^6| = |(b^3+1)^2 - (b^6+1)| \leq$$
$$\leq |(b^3+1)^2| + |b^6+1| = |b^3+1|^2 + |b^6+1| \leq$$
$$\leq 1^2 + 1 = 2 \Rightarrow 2|b^6| \leq 2 \Rightarrow |b|^6 \leq 1 \Rightarrow |b| \leq 1$$

By parallelogram identity:

$$|a+b|^2 + |a-b|^2 = 2(|a|^2 + |b|^2) \leq 2(1^2 + 1^2) = 4$$

□

Mathematics Department, National Economic College "Theodor Costescu", Drobeta Turnu - Severin, Romania
Email address: dansitaru63@yahoo.com