ROMANIAN MATHEMATICAL MAGAZINE

In ΔABC the following relationship holds:

$$\frac{a+b}{h_a}+\frac{b+c}{h_b}+\frac{c+a}{h_c}\ge 4\sqrt{3}$$

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Solution 1 by Khaled Abd Imouti, Kasem Abotrabi-Syria

$$\underbrace{\frac{a+b}{h_a}+\frac{b+c}{h_b}+\frac{c+a}{h_c}}_{l}=\underbrace{\left(\frac{a}{h_a}+\frac{b}{h_b}+\frac{c}{h_c}\right)}_{l_1}+\underbrace{\left(\frac{b}{h_a}+\frac{c}{h_b}+\frac{a}{h_c}\right)}_{l_2}$$

$$l_1=\frac{a^2}{a.h_a}+\frac{b^2}{b.h_b}+\frac{c^2}{c.h_c}=\frac{a^2}{2F}+\frac{b^2}{2F}+\frac{c^2}{2F}=\frac{a^2+b^2+c^2}{2F}\ \textbf{(I)}$$

In ΔABC and ΔA'B'C' : A'B'=B'C'=A'C' =1 (Daniel Pedoe 's)

$$a^2(1^2+1^2-1^2)+b^2(1^2+1^2-1^2)+C^2(1^2+1^2-1^2)\ge 16F\,.F'$$

$$F'=\frac{\sqrt{3}}{4}\Rightarrow a^2+b^2+c^2\ge 16F\times\frac{\sqrt{3}}{4}=4\sqrt{3}F\Rightarrow l_1\ge 2\sqrt{3}$$

$$l_2=\frac{b}{h_a}+\frac{c}{h_b}+\frac{a}{h_c}=\frac{ab}{a.h_a}+\frac{bc}{b.h_b}+\frac{ca}{c.h_c}=\frac{1}{\sin A}+\frac{1}{\sin B}+\frac{1}{\sin C}\qquad \textbf{(II)}$$

Let be the function $f(x)=\frac{1}{\sin x}$

By using Jensen's inequality:

$$\frac{1}{\sin A}+\frac{1}{\sin B}+\frac{1}{\sin C}\ge 3\left(\frac{1}{\sin\frac{A+B+C}{3}}\right)=2\sqrt{3}$$

$$l=l_1+l_2\ge 4\sqrt{3}$$

Solution 2 by Khaled Abd Imouti, Kasem Abotrabi-Syria

$$P_\Delta=a+b+c\ ,\quad (a+b+c)^2\ge 12\sqrt{3}S_\Delta$$

$$\sqrt{\frac{a^2+b^2+c^2}{3}}\ge\frac{a+b+c}{3}\quad \ldots.(QM-AM)\Rightarrow\frac{a^2+b^2+c^2}{3}\ge\frac{(a+b+c)^2}{9}\Rightarrow$$

$$a^2+b^2+c^2\ge\frac{(a+b+c)^2}{3}\Rightarrow a^2+b^2+c^2\ge 4\sqrt{3}S_\Delta\Rightarrow$$

$$\frac{a^2}{S_\Delta}+\frac{b^2}{S_\Delta}+\frac{c^2}{S_\Delta}=4\sqrt{3} \qquad (I)$$

$$l=\frac{a+b}{h_a}+\frac{b+c}{h_b}+\frac{c+a}{h_c}=\frac{a}{h_a}+\frac{b}{h_b}+\frac{c}{h_c}+\frac{b}{h_a}+\frac{c}{h_b}+\frac{a}{h_c}$$

$$=\frac{a^2}{a.h_a}+\frac{b^2}{b.h_b}+\frac{c^2}{c.h_c}+\frac{b}{h_a}+\frac{c}{h_b}+\frac{a}{h_c}$$

$$=\underbrace{\left(\frac{a^2}{2S_\Delta}+\frac{b^2}{2S_\Delta}+\frac{c^2}{2S_\Delta}\right)}_{l_1}+\underbrace{\left(\frac{b}{h_a}+\frac{c}{h_b}+\frac{a}{h_c}\right)}_{l_2}$$

$$l_2=\frac{ab}{a.h_a}+\frac{bc}{b.h_b}+\frac{ca}{c.h_c}=\frac{1}{2S_\Delta}(ab+bc+ca)=$$

$$=\frac{1}{2S_\Delta}\left(\frac{2S_\Delta}{\sin A}+\frac{2S_\Delta}{\sin B}+\frac{2S_\Delta}{\sin C}\right)=\frac{1}{\sin A}+\frac{1}{\sin B}+\frac{1}{\sin C}$$

Let be the function $f(x)=\frac{1}{\sin x}$

By using Jensen's inequality:

$$l_2\ge 2\sqrt{3} \Rightarrow l_2\ge 3\left(\frac{1}{\sin\frac{A+B+C}{3}}\right)$$

So : $l=l_1+l_2\ge 2\sqrt{3}+2\sqrt{3}=4\sqrt{3}$