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In ΔABC the following relationship holds:

$$\frac{a(b+c)}{a^2+bc}+\frac{b(c+a)}{b^2+ca}+\frac{c(a+b)}{c^2+ab}\geq 12\frac{r^2}{R^2}$$

Proposed by D.M.Bătinețu-Giurgiu, Claudia Nănuți-Romania

Solution by Tapas Das-India

$$abc=4Rrs\overset{Euler,Mitrinovic}{\geq}4.2.3\sqrt{3}r=\left(2\sqrt{3}r\right)^3\ (1)$$

$$:\frac{a(b+c)}{a^2+bc}+\frac{b(c+a)}{b^2+ca}+\frac{c(a+b)}{c^2+ab}\overset{AM-GM}{\geq}3\sqrt[3]{\frac{a(b+c)}{a^2+bc}\frac{b(c+a)}{b^2+ca}\frac{c(a+b)}{c^2+ab}}=$$

$$=3\sqrt[3]{\frac{abc(a+b)(c+a)(b+c)}{(a^2+bc)(b^2+ca)(c^2+ab)}}\overset{Cesaro\ \&\ AM-GM}{\geq}$$

$$\geq 3\sqrt[3]{\frac{abc8abc}{\frac{\left((a^2+bc)+(b^2+ca)+(c^2+ab)\right)^3}{27}}}=9\times\frac{2(abc)^{\frac{2}{3}}}{(a^2+b^2+c^2+ab+bc+ca)}\geq$$

$$\geq 9\times\frac{2(abc)^{\frac{2}{3}}}{(a^2+b^2+c^2+a^2+b^2+c^2)}\overset{Leibniz\ \&\ (1)}{\geq}\frac{9\times 2\times 12r^2}{2\times 9R^2}=12\frac{r^2}{R^2}$$

Equality holds for an equilateral triangle.