ROMANIAN MATHEMATICAL MAGAZINE

If $m \geq 0$ *then in triangle ABC the following inequality holds:*

$$\left(\frac{m^2a^4}{h_b^2h_c^2}+2\right)\left(\frac{m^2b^4}{h_a^2h_c^2}+2\right)\left(\frac{m^2c^4}{h_b^2h_a^2}+2\right) \geq 48m^2$$

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Let's use Arkady Alt's formula to prove:

$$x, y, z \geq 0\ ,\ \ t > 0$$

$$(x^2+t^2)(y^2+t^2)(z^2+t^2) \geq \frac{3}{4}t^4(x+y+z)^2$$

$$Here:\ \ x = \frac{ma^2}{h_bh_c}\ ,\qquad y = \frac{mb^2}{h_ah_c}\ ,\qquad z = \frac{mc^2}{h_ah_b}\ ,\qquad t = \sqrt{2}$$

$$\left(\frac{m^2a^4}{h_b^2h_c^2}+2\right)\left(\frac{m^2b^4}{h_a^2h_c^2}+2\right)\left(\frac{m^2c^4}{h_b^2h_a^2}+2\right) \overset{Arkady\ Alt}{\geq} \frac{3}{4}.4.\left(\frac{ma^2}{h_bh_c}+\frac{mb^2}{h_ah_c}+\frac{mc^2}{h_ah_b}\right)^2 \overset{A-G}{\geq}$$

$$\geq 3\left(3\left(\frac{m^3a^2b^2c^2}{h_a^2h_b^2h_c^2}\right)^{\frac{1}{3}}\right)^2 = 27m^2\left(\frac{abc}{h_ah_bh_c}\right)^{\frac{4}{3}} = 27m^2\left(\frac{4RF}{\frac{8F^3}{4RF}}\right)^{\frac{4}{3}} = 27m^2\left(\frac{2R^2}{F}\right)^{\frac{4}{3}} \geq$$

$$\geq 27m^2\left(2R^2.\frac{4}{\sqrt{27}R^2}\right)^{\frac{4}{3}} = 27m^2\left(\frac{8}{\sqrt{27}}\right)^{\frac{4}{3}} = 27m^2\left(\frac{2}{\sqrt{3}}\right)^4 = 48m^2$$

Equality holds for $a = b = c$, *and* $t = \sqrt{2}$.