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Let g_a, g_b, g_c be the lengths of Gergonne's cevians of the triangle ABC, M be an interior point in triangle and d_a, d_b, d_c the distances of point M to the sides BC, CA, AB respectively.

Prove that : $\left(\frac{g_a}{d_a}\right)^2 + \left(\frac{g_b}{d_b}\right)^2 + \left(\frac{g_c}{d_c}\right)^2 \geq 27$

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$$\left(\frac{g_a}{d_a}\right)^2 + \left(\frac{g_b}{d_b}\right)^2 + \left(\frac{g_c}{d_c}\right)^2 \geq \frac{1}{3}\left(\sum_{cyc}\frac{g_a}{d_a}\right)^2 \overset{g_a \geq h_a \text{ and analogs}}{\geq} \frac{1}{3}\left(\sum_{cyc}\frac{h_a}{d_a}\right)^2$$

$$= \frac{1}{3}\left(\sum_{cyc}\frac{2F}{ad_a}\right)^2 \overset{\text{Bergstrom}}{\geq} \frac{1}{3}\left(2F.\frac{9}{ad_a + bd_b + cd_c}\right)^2 = \frac{1}{3}\left(2F.\frac{9}{2F}\right)^2 = 27$$

$$\left(\begin{array}{l}\because [\Delta\, MBC] + [\Delta\, MCA] + [\Delta\, MAB] = \frac{1}{2}ad_a + \frac{1}{2}bd_b + \frac{1}{2}cd_c \\ \Rightarrow 2[\Delta\, ABC] = ad_a + bd_b + cd_c \Rightarrow 2F = ad_a + bd_b + cd_c\end{array}\right)$$

$$\therefore \left(\frac{g_a}{d_a}\right)^2 + \left(\frac{g_b}{d_b}\right)^2 + \left(\frac{g_c}{d_c}\right)^2 \geq 27\ \forall\ \Delta\, ABC, ''='' \text{ iff } \Delta\, ABC \text{ is equilateral (QED)}$$