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If w_a, w_b, w_c are the interior bisector lengths of triangle ABC with area F then

$$(w_a \cdot w_b + 2c^2)(w_b \cdot w_c + 2a^2)(w_c \cdot w_a + 2b^2) \geq 144\sqrt{3}F^3$$

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Due to Arkady Alt, if $X, Y, Z, T > 0$, then : $(X^2 + T^2)(Y^2 + T^2)(Z^2 + T^2) \geq \frac{3}{4} \cdot T^4 \cdot (X + Y + Z)^2$, with equality iff $X = Y = Z = \frac{T}{\sqrt{2}}$ and now,

$$\begin{aligned} & (w_a \cdot w_b + 2c^2)(w_b \cdot w_c + 2a^2)(w_c \cdot w_a + 2b^2) \\ &= a^2 b^2 c^2 \left(\frac{w_a w_b}{c^2} + 2 \right) \left(\frac{w_b w_c}{a^2} + 2 \right) \left(\frac{w_c w_a}{b^2} + 2 \right) \\ &= 16R^2 F^2 \cdot (X^2 + T^2)(Y^2 + T^2)(Z^2 + T^2) \\ & \left(X = \frac{\sqrt{w_a w_b}}{c}, Y = \frac{\sqrt{w_b w_c}}{a}, Z = \frac{\sqrt{w_c w_a}}{b}, T = \sqrt{2} \right) \stackrel{\text{Arkady Alt}}{\geq} \\ & 16R^2 F^2 \cdot \frac{3}{4} \cdot (\sqrt{2})^4 \cdot \left(\frac{\sqrt{w_a w_b}}{c} + \frac{\sqrt{w_b w_c}}{a} + \frac{\sqrt{w_c w_a}}{b} \right)^2 \\ & \geq 16R^2 F^2 \cdot 3 \left(\frac{\sqrt{h_a h_b}}{c} + \frac{\sqrt{h_b h_c}}{a} + \frac{\sqrt{h_c h_a}}{b} \right)^2 \stackrel{\text{AM-GM}}{\geq} 16R^2 F^2 \cdot 27 \cdot \left(\sqrt[3]{\frac{h_a h_b h_c}{abc}} \right)^2 \\ &= 16R^2 F^2 \cdot 27 \cdot \sqrt[3]{\frac{4F^4}{R^2 \cdot 16R^2 F^2}} = 16R^2 F^2 \cdot 27 \cdot \sqrt[3]{\frac{4F^3}{R^2 \cdot 16R^2 \cdot F}} \\ & \geq 16R^2 F^2 \cdot 27 \cdot \sqrt[3]{\frac{4F^3}{R^2 \cdot 16R^2 \cdot \frac{3\sqrt{3}R^2}{4}}} \\ & \left(\begin{array}{l} \text{Mitrinovic} \\ \text{and} \\ \text{Euler} \end{array} \right) \because R^4 \geq \frac{4s^2}{27} \cdot 4r^2 = \frac{16F^2}{27} \Rightarrow R^2 \geq \frac{4F}{3\sqrt{3}} \Rightarrow F \leq \frac{3\sqrt{3}R^2}{4} = \frac{16R^2 F^3 \cdot 27}{R^2 \cdot \sqrt{3}} \\ &= 144\sqrt{3} \cdot F^3 \therefore (w_a \cdot w_b + 2c^2)(w_b \cdot w_c + 2a^2)(w_c \cdot w_a + 2b^2) \geq 144\sqrt{3} \cdot F^3 \end{aligned}$$

$\forall \Delta ABC, " = " \text{ iff } a = b = c = 1 \text{ (QED)}$