ROMANIAN MATHEMATICAL MAGAZINE

If in triangle $T = ABC$ with the area F we denote

$$u(T) = \frac{a}{h_a}.\cot(A) + \frac{b}{h_b}.\cot(B) + \frac{c}{h_c}.\cot(C) \text{ then prove that:}$$

$$\left(a^2b^2 + u(T)\right)\left(c^2b^2 + u(T)\right)\left(a^2c^2 + u(T)\right) \geq 144F^2$$

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Solution by Mirsadix Muzefferov-Azerbaijan

$$u(T) = \frac{a}{h_a}.\cot(A) + \frac{b}{h_b}.\cot(B) + \frac{c}{h_c}.\cot(C) =$$

$$\frac{a}{\frac{2F}{a}}\cdot\frac{\cos(A)}{\sin(A)} + \frac{b}{\frac{2F}{b}}\cdot\frac{\cos(B)}{\sin(B)} + \frac{c}{\frac{2F}{c}}\cdot\frac{\cos(C)}{\sin(C)} =$$

$$\frac{(2R.\sin(A)).a\cos(A)}{2F.\sin(A)} + \frac{(2R.\sin(B)).b\cos(B)}{2F.\sin(B)} + \frac{(2R.\sin(C)).c\cos(C)}{2F.\sin(C)} =$$

$$\frac{R}{F}(a\cos(A) + b\cos(B) + c\cos(C)) = \frac{R}{F}.\frac{2F}{R} = 2$$

$$u(T) = 2$$

$$\left(a^2b^2 + u(T)\right)\left(c^2b^2 + u(T)\right)\left(a^2c^2 + u(T)\right) =$$

$$(a^2b^2 + 2)(c^2b^2 + 2)(a^2c^2 + 2) \overset{Arkady\ Alt}{\geq} \frac{3}{4}.(2)^2(ab + bc + ac)^2 \geq$$

$$\geq 3\left(4\sqrt{3}F\right)^2 = 144F^2 \ (Proved)$$

$$Equality\ holds\ for: \ a = b = c = 1, \quad u(T) = 2$$