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If a, b, c, x, y be > 0 *then* :

$$\left(\frac{a^2}{(bx+cy)^2}+1\right)\left(\frac{b^2}{(cx+ay)^2}+1\right)\left(\frac{c^2}{(ax+by)^2}+1\right)\ge\frac{27}{4(x+y)^2}$$

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Due to Arkady Alt, if $X, Y, Z, T > 0$, *then* :

$$(X^2+T^2)(Y^2+T^2)(Z^2+T^2)\ge\frac{3}{4}.T^4.(XY+YZ+ZX)^2,$$

with equality iff $X = Y = Z = \frac{1}{\sqrt{2}}$ and now,

$$\left(\frac{a^2}{(bx+cy)^2}+1\right)\left(\frac{b^2}{(cx+ay)^2}+1\right)\left(\frac{c^2}{(ax+by)^2}+1\right)=$$

$$(X^2+T^2)(Y^2+T^2)(Z^2+T^2)\ \left(X=\frac{a}{bx+cy}, Y=\frac{b}{cx+ay}, Z=\frac{c}{ax+by}, T=1\right)$$

$$\overset{\text{Arkady Alt}}{\ge}\frac{3}{4}.(1)^4.\left(\frac{a}{bx+cy}+\frac{b}{cx+ay}+\frac{c}{ax+by}\right)^2$$

$$=\frac{3}{4}.\left(\frac{a^2}{abx+cay}+\frac{b^2}{bcx+aby}+\frac{c^2}{cax+bcy}\right)^2\overset{\text{Bergstrom}}{\ge}$$

$$\frac{3}{4}.\left(\frac{(a+b+c)^2}{(ab+bc+ca)(x+y)}\right)^2\ge\frac{3}{4}.\left(\frac{3(ab+bc+ca)}{(ab+bc+ca)(x+y)}\right)^2$$

$$\therefore\left(\frac{a^2}{(bx+cy)^2}+1\right)\left(\frac{b^2}{(cx+ay)^2}+1\right)\left(\frac{c^2}{(ax+by)^2}+1\right)\ge\frac{27}{4(x+y)^2}$$

$\forall\, a, b, c, x, y > 0$, " = " iff $\frac{a}{bx+cy}=\frac{b}{cx+ay}=\frac{c}{ax+by}=\frac{1}{\sqrt{2}}$ and $a = b = c$

and so, " = " iff $x + y = \sqrt{2} \wedge a = b = c$ (QED)