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If $a, b, c, u, v, w > 0$, *then* :

$$(a^2+2u^2)(b^2+2v^2)(c^2+2w^2) \geq 3(avw+buw+cuv)^2$$

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Due to Arkady Alt, if $x, y, z, T > 0$, *then* :

$$(x^2+T^2)(y^2+T^2)(z^2+T^2) \geq \frac{3}{4}.T^4.(xy+yz+zx)^2,$$

$$\text{with equality iff } x = y = z = \frac{1}{\sqrt{2}} \to (1)$$

$$(a^2+2u^2)(b^2+2v^2)(c^2+2w^2) =$$

$$= \frac{(2a^2v^2w^2+4u^2v^2w^2)(2b^2u^2w^2+4u^2v^2w^2)(2c^2u^2v^2+4u^2v^2w^2)}{8(uvw)^4} =$$

$$= \frac{(x^2+T^2)(y^2+T^2)(z^2+T^2)}{8(uvw)^4} \left(x = \sqrt{2}.avw, y = \sqrt{2}.buw, z = \sqrt{2}.cuv, T = 2uvw\right) \geq$$

$$\overset{\text{Arkady Alt}}{\geq} \frac{\frac{3}{4}.T^4.(xy+yz+zx)^2}{8(uvw)^4} = \frac{\frac{3}{4}.(2uvw)^4.\left(\sqrt{2}.avw+\sqrt{2}.buw+\sqrt{2}.cuv\right)^2}{8(uvw)^4}$$

$$\therefore (a^2+2u^2)(b^2+2v^2)(c^2+2w^2) \geq 3(avw+buw+cuv)^2 \ \forall\, a, b, c, u, v, w > 0,$$

$$'' ='' \text{ iff } avw = buw = cuv = \frac{1}{2} \text{ (QED)}$$