DANIEL SITARU MARIAN URSĂRESCU

OLYMPIAD PROBLEMS ALGEBRA VOLUME II

ABOUT AUTHORS

Daniel Sitaru, born on 9 August 1963 in Craiova, Romania, is a teacher at National Economic College "Theodor Costescu" in Drobeta Turnu Severin. He published 43 mathematical books, last five of these "Math Phenomenon", "Algebraic Phenomenon", "Analytical Phenomenon", "The Olympic Mathematical Marathon" and "699 Olympic Mathematical Challenges" (the last one with Mihály Bencze), were very appreciated world wide. He is the founding editor of "Romanian Mathematical Magazine", an Interactive Mathematical Journal with 5.600.000 visitors, in the last three years (www.ssmrmh.ro).Many problems from his books were published in famous journals such as "American Mathematical Monthly", "Crux Mathematicorum", "Math Problems Journal", "The Pentagon Journal", La Gaceta de la RSME", "SSMA Magazine". He also published an impressive number of original problems in all mathematical journals from Romania (GMB, Cardinal, Elipsa, Argument, Recreații Matematice). His articles from "Crux Mathematicorum" and "The Pentagon Journal" were also very appreciated.

Marian Ursărescu, was born on 1st of June 1965, in Focșani. He graduated from A.I. Cuza University, Faculty of Mathematics, in 1988. He is a teacher of mathematics from 1988 at "Roman Vodă" National Colledge in Roman. Starting from 1990 until now, he had 47 pupils that participated on the Mathematical National Olympiad, which from 28 had obtained prizes and Olympic mentions.

He published over 100 problems and articles in Mathematical National Gazette . Also, he published several problems and articles in mathematical magazines such as "Mathematical Recreations", "Romanian Mathematical Magazine", "Let's understand math." A lot of his proposed problems had been selected in various mathematical contests, olympiads and mathematical books. He co-authored "Functional Equations" together with M. O. Drâmbe and another 5 books with Mihaly Bencze and Daniel Sitaru.

FROM AUTHORS

In July 2016 was founded "Romanian Mathematical Magazine" (RMM) (www.ssmrmh.ro) as an Interactive Mathematical Journal.

Same date was founded "Romanian Mathematical Magazine"-Online Mathematical Journal (ISSN-2501-0099) and "Romanian Mathematical Magazine"-Paper Variant (ISSN-1584-4897).

In three years the website of RMM was visited by over 5,000,000 people from all over the world. With over 10,000 proposed problems posted, over 14,000 solutions and many math articles and math notes RMM is a big chance for young mathematicians from whole world to be known as great proposers and solvers. This book is a small part of RMM-Interactive Journal.

Many thanks to RMM-Team for proposed problems and solutions.

CONTENT

EQUATIONS

1.1 Solve the equation in $\mathbb{R}$:

$$\sqrt{x^3-2x^2+2x}+3\sqrt[3]{x^2-x+1}+2\sqrt[4]{4x-3x^4}=\frac{x^4-3x^3}{2}+7$$

Hoang Le Nhat Tung

Solution (Amit Dutta)

$$\textit{Domain} \to \begin{cases} x^3-2x^2+2x>0 \\ 4x-3x^4>0 \end{cases}$$

$$x^3-2x^2+2x=x(x^2-2x+2)=x[(x-1)^2+1]$$

$$\because x^3-2x^2+2x>0 \Rightarrow x[(x-1)^2+1]>0 \Rightarrow x>0$$

$$GM \le AM \ \sqrt{x^2-2x^2+2x} \le \frac{(x^2-2x^2+2x)+1}{2}$$

$$\sqrt{x^3-2x^2+2x} \le \left(\frac{x^2-2x^2+2x+1}{2}\right) \quad (a)$$

Equality holds when $x^2-2x^2+2x=1$ *(1)*

Again, using $GM \le AM$

$$3\sqrt[3]{x^2-x+1} \le (x^2-x+1)+1+1 \le (x^2-x+3) \quad (2)$$

Equality holds when $x^2-x+1=1$ *(2)*

Again, using $GM \le AM$

$$2\sqrt[4]{4x-3x^4} \le 2\left\{\frac{(4x-3x^4)+1+1+1}{4}\right\} \le \left(\frac{4x-3x^4+3}{2}\right) \quad (3)$$

Equality holds when $4x-3x^4=1$ *(3)*

Adding (1), (2), (3):

$$\sqrt{x^3-2x^2+2x}+3\sqrt[3]{x^2-x+1}+2\sqrt[4]{4x-3x^4} \le$$

$$\le \left(\frac{x^3-2x^2+2x+1}{2}\right)+(x^2-x+3)+\left(\frac{4x-3x^4+3}{2}\right)$$

$$\Rightarrow \frac{x^4-3x^3}{2}+7 \le \frac{-3x^4+4x+10+x^3}{2}$$

$$\Rightarrow x^4-3x^3+14 \le -3x^4+4x+10+x^3$$

$$\Rightarrow 4x^4 - 4x^3 - 4x + 4 \le 0$$

$$\Rightarrow x^4 - x^3 - x + 1 \le 0 \Rightarrow x^3(x-1) - 1(x-1) \le 0$$

$$\Rightarrow (x^3 - 1)(x-1) \le 0 \Rightarrow (x-1)(x^2+x+1)(x-1) \le 0$$

$$\Rightarrow (x-1)^2(x^2+x+1) \le 0$$

$$\because x^2 + x + 1 = \left(x + \frac{1}{2}\right)^2 + \frac{3}{4} > 0 \Rightarrow (x-1)^2 \le 0$$

$$(x-1)^2 = 0 \Rightarrow x = 1 \quad \textit{(4)}$$

From (1), (2), (3) & (4): The only real solution is $x = 1$.

1.2 Find all real roots of the following equation:

$$(x^3 - 2)^3 + (x^2 - 2)^2 = 0$$

Nguyen Viet Hung

Solution (Orlando Irahola Ortega)

$$(2 - x^3)^3 = (x^2 - 2)^2$$

$$a = 2 - x^3 \Rightarrow a - 2 = -x^3 \dots (\,)^2 \Rightarrow (a-2)^2 = x^6 \quad \textit{(2)}$$

$$b = x^2 - 2 \Rightarrow b + 2 = x^2 \dots (\,)^3 \Rightarrow (b+2)^3 = x^6 \quad \textit{(3)}$$

$$a^3 = b^2 \quad \textit{(1)}$$

$$\textit{(2)=(3)}\Rightarrow \begin{cases} (b+2)^3 = (a-2)^2 \dots (4) \\ a^3 = b^2 \dots (1) \end{cases}$$

$$\textit{(4)}-\textit{(1)}:\Rightarrow (b+2)^3 - a^3 = (a-2)^2 - b^2$$

$$\Rightarrow (b + 2 - a)(a^2 + b^2 + ab + 4b + 2a + 4) = (b + 2 - a)(2 - a - b)$$

$$(\underbrace{b + 2 - a = 0}_{(A)} \wedge \underbrace{a^2 + ab + b^2 + 3a + 5b + 2 =}_{(B)} 0)$$

$$\textit{(A)}\ b + 2 = a \Rightarrow x^3 + x^2 - 2 = 0 \rightarrow (x-1)(x^2 + 2x + 2) = 0 \Rightarrow$$

$$x - 1 = 0 \Rightarrow x_1 = 1, \qquad x^2 + 2x + 2 = 0, \qquad x_{2,3} = -1 \pm i$$

$$\textit{(B)}\ a^2 + ab + b^2 + 3a + 5b + 2 = 0 \Rightarrow x^6 - \underbrace{\underbrace{x^5}_{+} + x^4 - \underbrace{3x^3}_{+}}_{(\forall)x\in\mathbb{R}} + 3x^2 + 2 = 0$$

$$x^6 - x^5 + x^4 - 3x^3 + 3x^2 + 2 > 0 \Rightarrow x \notin \mathbb{R}$$

$$C.S. = \{1\}$$

1.3 Find all real roots of the following equation:

$$\sqrt[2n]{2-x^2}+\sqrt[2n]{2|x|-1}=(x^2-1)^{2m}+2$$

where m, n are positive integers.

Nguyen Viet Hung

Solution (Michael Sterghiou)

$$\sqrt[2n]{2-x^2}+\sqrt[2n]{2|x|-1}=(x^2-1)^{2m}+2 \quad (1)$$

Let $y=|x|\geq 0$ *(1)* $\rightarrow \sqrt[2n]{2-y^2}+\sqrt[2n]{2y-1}=(y^2-1)^{2m}+2$ *(1)'*

Consider the function $f(t)=\sqrt[2n]{t}, t\geq 0$ *with* $f''(t)=\frac{(1-2n)t^{\frac{1}{2n}-2}}{4n^2}<0$

for $n\in\mathbb{N}, n\geq 1$*, hence* $f(t)$ *concave and from Jensen:*

LHS of (1) $\leq 2\cdot\sqrt[2n]{\frac{2-y^2+2y-1}{2}}=2\sqrt[2n]{\frac{1}{2}(-y^2+2y+1)}$ *(2)*

From (1)' we have $2-y^2\geq 0\rightarrow y\leq\sqrt{2}$ *and* $2y-1\geq 0\rightarrow y\geq\frac{1}{2}$ *or*

$\frac{1}{2}\leq y\leq\sqrt{2}$*. Now,* $-y^2+2y+1>0$ *and equality in (2) when*

$2-y^2=2y-1\leftrightarrow y=1$*. From (1) and (2)*

$\rightarrow 2\cdot\sqrt[2n]{\frac{1}{2}(-y^2+2y+1)}\geq(y^2-1)^{2m}+2$ *(3). Consider the function*

$f(y)=\frac{1}{2}(-y^2+2y+1)$ *with* $f'(y)=-y+1$ *with root* $y=1$ *and*

$f''(y)<0$ *with* $\max f=1$ *at* $y=1$*. As* $\frac{1}{2}(-y^2+2y+1)\leq 1\rightarrow$

$$\sqrt[2n]{\frac{1}{2}(-y^2+2y+1)}\leq 1 \text{ and}$$

$2\cdot\sqrt[2n]{\frac{1}{2}(-y^2+2y+1)}\leq 2$*. The last inequality and (3) give*

$2\geq$ *LHS of (3)* $\geq 2+(y^2-1)^{2m}$ *which can happen only if* $y^2-1=0$ *or*

$y=1$ *or* $|x|=1$ *or* $x=\pm 1$ *which are the only real solution of (1).*

1.4 Solve for real numbers ($a \geq 0$;fixed):

$$\sqrt[3]{3x^2-3x+1}+4\sqrt[4]{4x^3-3x^4}=ax^5+(1-5a)x+4a+4$$

Marin Chirciu

Solution (Michael Stergiou)

$$\sqrt[3]{3x^2-3x+1}+4\sqrt[4]{4x^3-3x^4}=ax^5+(1-5a)x+4a+4 \quad (1)$$

$ax^5+a+a+a+a \geq 5\cdot\sqrt[5]{a^5x^5}=5ax.$ *So, RHS of (1)*$\geq x+4 \quad (2)$

$$\sqrt[3]{(3x^2-3x+1)\cdot 1\cdot 1} \overset{Am-Gm}{\leq} \frac{3x^2-3x+1+1+1}{3}=x^2-x+1 \quad (3)$$

$$4\sqrt[4]{(4x^3-3x^4)\cdot 1\cdot 1\cdot 1} \overset{Am-Gm}{\leq} 4\cdot\frac{4x^3-3x^4+1+1+1}{4}$$

$$=4x^3-3x^4+3 \quad (4)$$

Therefore LHS of (1)$\leq x^2-x+1+4x^3-3x^4+3$ *while RHS of (1)* $\geq x+4$

But: $x^2-x+1+4x^3-3x^4+3-(x+4)=-x(x-1)^2(3x+2)\leq 0$

Hence we can have only equalities for x=1.

1.5 Find $x,y,z,t>0$ such that:

$$(1+x)^5(1+y)^4(1+z)^3(1+t)^2=3125xyzt$$

Daniel Sitaru

Solution

$$\frac{x}{1+x}+\frac{y}{(1+x)(1+y)}+\frac{z}{(1+x)(1+y)(1+z)}+\frac{t}{(1+x)(1+y)(1+z)(1+t)}+$$

$$+\frac{1}{(1+x)(1+y)(1+z)(1+t)}=$$

$$=\frac{1+x-1}{1+x}+\frac{1+y-1}{(1+x)(1+y)}+\frac{1+z-1}{(1+x)(1+y)(1+z)}+\frac{1+t-1}{(1+x)(1+y)(1+z)(1+t)}-$$

$$+\frac{1}{(1+x)(1+y)(1+z)(1+t)}=$$

$$=\left(1-\frac{1}{1+x}\right)+\left(\frac{1}{1+x}-\frac{1}{(1+x)(1+y)}\right)+\left(\frac{1}{(1+x)(1+y)}-\frac{1}{(1+x)(1+y)(1+z)}\right)+$$

$$+\left(\frac{1}{(1+x)(1+y)(1+z)}-\frac{1}{(1+x)(1+y)(1+z)(1+t)}\right)+$$
$$+\frac{1}{(1+x)(1+y)(1+z)(1+t)}=1$$
$$1=\frac{x}{1+x}+\frac{y}{(1+x)(1+y)}+\frac{z}{(1+x)(1+y)(1+z)}$$
$$+\frac{t}{(1+x)(1+y)(1+z)(1+t)}+$$
$$+\frac{1}{(1+x)(1+y)(1+z)(1+t)}\ge$$
$$\overset{AM-GM}{\ge}5\cdot\sqrt[5]{\frac{x}{1+x}\cdot\frac{y}{(1+x)(1+y)}\cdot\frac{z}{(1+x)(1+y)(1+z)}\cdot\frac{t}{(1+x)(1+y)(1+z)(1+t)}\cdot\frac{1}{(1+x)(1+y)(1+z)(1+t)}}=$$
$$=5\cdot\sqrt[5]{\frac{xyzt}{(1+x)^5(1+y)^4(1+z)^3(1+t)^2}}$$
$$1\ge 5^5\cdot\frac{xyzt}{(1+x)^5(1+y)^4(1+z)^3(1+t)^2}$$
$$(1+x)^5(1+y)^4(1+z)^3(1+t)^2\ge 3125xyzt$$

Equality in AM-GM holds if:

$$\frac{x}{1+x}=\frac{y}{(1+x)(1+y)}=\frac{z}{(1+x)(1+y)(1+z)}=\frac{t}{(1+x)(1+y)(1+z)(1+t)}=$$
$$=\frac{1}{(1+x)(1+y)(1+z)(1+t)}$$
$$t=1;\frac{z}{(1+x)(1+y)(1+z)}=\frac{1}{(1+x)(1+y)(1+z)\cdot 2}\Rightarrow z=\frac{1}{2}$$
$$\frac{y}{(1+x)(1+y)}=\frac{\frac{1}{2}}{(1+x)(1+y)\left(1+\frac{1}{2}\right)}\Rightarrow y=\frac{\frac{1}{2}}{\frac{3}{2}}=\frac{1}{3}$$
$$\frac{x}{1+x}=\frac{\frac{1}{3}}{(1+x)\left(1+\frac{1}{3}\right)}\Rightarrow x=\frac{\frac{1}{3}}{1+\frac{1}{3}}=\frac{\frac{1}{3}}{\frac{3+1}{3}}=\frac{\frac{1}{3}}{\frac{4}{3}}=\frac{1}{4}$$

Solution: $x=\frac{1}{4};y=\frac{1}{3};z=\frac{1}{2};t=1$

1.6 Solve for real numbers:

$$\mathbf{2\sqrt{\frac{x-2}{x+3}}+\frac{x+4}{\left(\sqrt{x-2}+\sqrt{x+4}\right)^2}=1}$$

Hoang Le Nhat Tung

Solution (Hoang Le Nhat Tung)

We have: $x \geq 2$

Let (1): $2\sqrt{\frac{x-2}{x+3}} + \frac{x+4}{(\sqrt{x-2}+\sqrt{x+4})^2} = 1 \Leftrightarrow 2\sqrt{\frac{x-2}{x+3}} + \frac{1}{\frac{(\sqrt{x-2}+\sqrt{x+4})^2}{x+4}} = 1 \Leftrightarrow$

$$2\sqrt{\frac{x-2}{x+3}} + \frac{1}{\left(\sqrt{\frac{x-2}{x+4}} + \sqrt{\frac{x+4}{x+4}}\right)^2} = 1 \Leftrightarrow 2\sqrt{\frac{x-2}{x+3}} + \frac{1}{\left(\sqrt{\frac{x-2}{x+4}} + 1\right)^2} = 1; (2)$$

Because $x \geq 2 > 0$ *then* $x - 2 \geq 0$. *Hence* $0 < x + 3 < x + 4 \Rightarrow$

$$\frac{1}{x+3} > \frac{1}{x+4} > 0 \Rightarrow \frac{x-2}{x+3} \geq \frac{x-2}{x+4} \Rightarrow \sqrt{\frac{x-2}{x+3}} \geq \sqrt{\frac{x-2}{x+4}}$$

$$\Leftrightarrow 2\sqrt{\frac{x-2}{x+3}} + \frac{1}{\left(\sqrt{\frac{x-2}{x+4}} + 1\right)^2} \geq 2\sqrt{\frac{x-2}{x+4}} + \frac{1}{\left(\sqrt{\frac{x-2}{x+4}} + 1\right)^2} \Leftrightarrow$$

$$2\sqrt{\frac{x-2}{x+3}} + \frac{1}{\left(\sqrt{\frac{x-2}{x+4}}+1\right)^2} \geq 2t + \frac{1}{(t-1)^2}; \textit{let } t = \sqrt{\frac{x-2}{x+4}} \geq 0; (3)$$

By AM-GM inequality for three positive real numbers, we have:

$$2t + \frac{1}{(t+1)^2} = (t+1) + (t+1) + \frac{1}{(t+1)^2} - 2$$

$$\geq 3\sqrt[3]{(t+1)(t+1) \cdot \frac{1}{(t+1)^2}} - 2 = 3\sqrt[3]{1} - 2 = 1 \Rightarrow$$

$$2t + \frac{1}{(t-1)^2} \geq 1. \textit{Let with (3)} \Rightarrow 2\sqrt{\frac{x-2}{x+3}} + \frac{1}{\left(\sqrt{\frac{x-2}{x+4}}+1\right)^2} \geq 1; (4)$$

Let (2),(4)$\Rightarrow 2\sqrt{\frac{x-2}{x+3}} + \frac{1}{\left(\sqrt{\frac{x-2}{x+4}}+1\right)^2} = 1$ *and equality occurs if* $\begin{cases} x - 2 = 0 \\ t + 1 = \frac{1}{(t+1)^2} \end{cases} \Leftrightarrow$

$$\begin{cases} x = 2 \\ \sqrt{\frac{x-2}{x+4}} = 0 \end{cases} \Leftrightarrow x = 2. \textit{The solution of equation is } S = \{2\}$$

1.7 Let be: $\boldsymbol{A = \left\{ x \mid x \in \mathbb{R}, \sqrt[7]{2+x} + \sqrt[7]{5-x} = \sqrt[7]{7} \right\}}$

$\boldsymbol{B = \left\{ x \mid x \in \mathbb{R}, \sqrt[9]{3+x} + \sqrt[9]{6-x} = \sqrt[9]{9} \right\}}$**. Find the sets** $\boldsymbol{\Omega_1, \Omega_2}$ **such that:** $\boldsymbol{A \Delta \Omega_1 = B \quad \Omega_2 \Delta B = A, \quad \left(X \Delta Y = (X/Y) \cup (Y/X)\right)}$

Daniel Sitaru

Solution (Ravi Prakash)

$$\sqrt[7]{2+x} + \sqrt[7]{5-x} = \sqrt[7]{7}$$

$$\textit{Put: } \sqrt[7]{2+x} = X, \sqrt[7]{5-x} = Y \Rightarrow 2 + x = X^7, 5 - x = Y^7 \Rightarrow X^7 + Y^7 = 7$$

$$\textit{Also, } X + Y = \sqrt[7]{7} \Rightarrow (X+Y)^7 = 7 = X^7 + Y^7 \Rightarrow$$

$$\binom{7}{1} X^6 Y + \binom{7}{2} X^5 Y^2 + \binom{7}{3} X^4 Y^3 + \binom{7}{4} X^3 Y^4 + \binom{7}{5} X^2 Y^5 + \binom{7}{6} X Y^6 = 0$$

$$7XY[(X^5 + Y^5) + 3XY(X^3 + Y^3) + 5X^2Y^2(X+Y)] = 0$$

$$7XY(X+Y)[X^4 - X^3Y + X^2Y^2 - XY^3 + Y^4 + 3X^3Y - 3X^2Y^2 + 3XY^3 + 5X^2Y^2] = 0$$

$$XY[X^4 - 2X^3Y + 3X^2Y^2 + 2XY^3 + Y^4] = 0$$

$$XY(X^2 + XY + Y^2)^2 = 0 \textit{ how } X^2 + XY + Y^2 > 0 \Rightarrow XY = 0 \Rightarrow X = 0 \textit{ or}$$

$$Y = 0 \Rightarrow x \in \{-2,5\} = A. \textit{ Similarly,}$$

$$\sqrt[9]{3+x} + \sqrt[9]{6-x} = \sqrt[9]{9} \Rightarrow x \in \{-3,6\} = B$$

$$A\Delta\Omega_1 = B \Rightarrow (\{-2,5\} / \Omega_1) \cup (\Omega_1/\{-2,5\}) = \{-3,6\} \Rightarrow \Omega_1 = \{-2,5,-3,6\}$$

$$\Omega_2\Delta B = A \Rightarrow (\{-3,6\}/\Omega_2) \cup (\Omega_2/\{-3,6\}) = \{-2,5\} \Rightarrow \Omega_2 = \{-2,5,3,6\}$$

1.8 Solve for real numbers:

$$\begin{cases} \boldsymbol{\sqrt{x^2+y^2} + \sqrt{(x-4)^2+y^2} + \sqrt{x^2+(y-3)^2} + \sqrt{(x-4)^2+(y-3)^2} = 10} \\ \boldsymbol{x + 2y = 5z} \end{cases}$$

Daniel Sitaru

Solution (Abner Chinga Bazo)

By Minkowsky inequality we have:

$$\sqrt{x^2+y^2}+\sqrt{(x-4)^2+(y-3)^2} \geq \sqrt{3^2+4^2}=5$$

Equality occurs when: $\frac{x-4}{x}=\frac{y-3}{y}=k \Rightarrow \begin{cases} x=4k \\ y=3k \end{cases}$

If $\sqrt{x^2+y^2}+\sqrt{(x-4)^2+(y-3)^2}=5$

$$\Rightarrow \sqrt{(3k)^2+(4k)^2}+\sqrt{(3k-4)^2+(4k-3)^2}=5$$

$$\Rightarrow \sqrt{25k^2}+\sqrt{25(k^2-2k+1)}=5 \Rightarrow 5|k|+5|k-1|=5$$

$$\Rightarrow |k|+|k-1|=1$$

Case 1: $k \in (-\infty, 0) \Rightarrow -k-(k-1)=1 \Rightarrow k=0 \Rightarrow k \in \emptyset$

Case 2: $k \in [0,1) \Rightarrow k-(k-1)=1 \Rightarrow k \in \mathbb{R} \Rightarrow k \in [0,1)$

Case 3: $k \in [1,\infty) \Rightarrow k+k-1=1 \Rightarrow k=1$

$$\sqrt{x^2+y^2}+\sqrt{(x-4)^2+(y-3)^2}=5 \Rightarrow k \in [0,1) \quad (i)$$

$$\sqrt{(x-4)^2+y^2}+\sqrt{x^2+(y-3)^2} \geq \sqrt{3^2+4^2}=5$$

Equality occurs when: $\frac{x-4}{x}=\frac{y-4}{y}=k \Rightarrow \begin{cases} x=4k \\ y=3k \end{cases}$

If $\sqrt{(x-4)^2+y^2}+\sqrt{x^2+(y-3)^2}=5$

$$\Rightarrow \sqrt{(4k-4)^2+(3k)^2}+\sqrt{(4k)^2+(3k-3)^2}=5$$

$\Rightarrow \sqrt{25k^2-32k+16}+\sqrt{25k^2-18k+9}=5$ *solving the equation:* $k=\frac{1}{2}$

$$\sqrt{(x-4)^2+y^2}+\sqrt{x^2+(y-3)^2}=5 \Rightarrow k=\frac{1}{2} \quad (ii)$$

From (i),(ii) we get:

$$\sqrt{x^2+y^2}+\sqrt{(x-4)^2+y^2}+\sqrt{(x-4)^2+(y-3)^2}=10,$$

$$x=4k, y=3k, k=\frac{1}{2}$$

If $\sqrt{x^2+y^2}+\sqrt{(x-4)^2+y^2}+\sqrt{(x-4)^2+(y-3)^2}=10 \Rightarrow x=2; y=\frac{3}{2}$

But: $x+2y=5z \Rightarrow z=1.$

$$\textit{So, } (x,y,z)=\left(2,\tfrac{3}{2},1\right)$$

1.9 Solve for real numbers:

$$\begin{cases} \sin x = \cos y \\ \begin{vmatrix} \sin(x+y) & \sin(y+\sqrt{xy}) & \sin(\sqrt{xy}+x) \\ \cos(x+y) & \cos(y+\sqrt{xy}) & \cos(\sqrt{xy}+x) \\ \cos(x-y) & \cos(y-\sqrt{xy}) & \cos(\sqrt{xy}-x) \end{vmatrix} = 0 \end{cases}$$

Daniel Sitaru

Solution (Adrian Popa)

$$\Delta \to \begin{vmatrix} \sin(x+y) & \sin(y+\sqrt{xy}) & \sin(\sqrt{xy}+x) \\ \cos(x+y) & \cos(y+\sqrt{xy}) & \cos(\sqrt{xy}+x) \\ \cos(x-y) & \cos(y-\sqrt{xy}) & \cos(\sqrt{xy}-x) \end{vmatrix} = 0$$

$$\sin x = \cos y \Rightarrow \sin x = \sin\left(\frac{\pi}{2}-y\right)$$

$$\sin(x+y) = \sin x\cos y + \sin y\cos x = \cos^2 y + \sin^2 y = 1 \quad (1)$$

$$\therefore \sin^2 x = \cos^2 y$$

$$1-\cos^2 x = \cos^2 y \Rightarrow \cos^2 x = 1-\cos^2 y \Rightarrow \cos x = \sin y$$

$$\cos(x+y) = \cos x\cos y - \sin x\sin y = \sin y\cos y - \cos y\sin y = 0 \quad (2)$$

$$\cos(x-y) = \cos x\cos y + \sin x\sin y = \sin y\cos y + \cos y\sin y =$$

$$2\sin y\cos y = \sin 2y \quad (3)$$

$$\Delta = \begin{vmatrix} 1 & \sin(y+\sqrt{xy}) & \sin(\sqrt{xy}+x) \\ 0 & \cos(y+\sqrt{xy}) & \cos(\sqrt{xy}+x) \\ \sin 2y & \cos(y-\sqrt{xy}) & \cos(\sqrt{xy}-x) \end{vmatrix} = 0$$

We develop after the first column:

$$\cos(y+\sqrt{xy})\cdot\cos(\sqrt{xy}-x) - \cos(\sqrt{xy}+x)\cdot\cos(y-\sqrt{xy}) +$$

$$+\sin 2y\left(\sin(y+\sqrt{xy})\cdot\cos(\sqrt{xy}+x) - \sin(\sqrt{xy}+x)\cdot\cos(y+\sqrt{xy})\right) = 0 \Rightarrow$$

$$\Rightarrow \cos(y+\sqrt{xy})\cdot\cos\left(\sqrt{xy}-\frac{\pi}{2}+y\right) - \cos\left(\sqrt{xy}+\frac{\pi}{2}-y\right)\cdot\cos(y-\sqrt{xy}) +$$

$$+\sin 2y\left(\sin\left(y+\sqrt{xy}\right)\cdot\cos\left(\sqrt{xy}+\frac{\pi}{2}-y\right)-\sin\left(\sqrt{xy}+\frac{\pi}{2}-y\right)\cdot\cos\left(y+\sqrt{xy}\right)\right)=0$$

$$\cos\left(y+\sqrt{xy}\right)\cdot\sin\left(y+\sqrt{xy}\right)-\sin\left(y-\sqrt{xy}\right)\cdot\cos\left(y-\sqrt{xy}\right)+$$

$$+\sin 2y\left(\sin\left(y+\sqrt{xy}\right)\cdot\sin\left(y-\sqrt{xy}\right)-\cos\left(y-\sqrt{xy}\right)\cdot\cos\left(y+\sqrt{xy}\right)\right)=0\Rightarrow$$

$$\Rightarrow\frac{\sin 2\left(y+\sqrt{xy}\right)}{2}-\frac{\sin 2\left(y-\sqrt{xy}\right)}{2}-\sin 2y\left(\cos 2y\right)=0\Rightarrow$$

$$\Rightarrow\frac{2\sin 2\sqrt{xy}\cos 2y}{2}-\sin 2y\cos 2y=0$$

$$\cos 2y\left(\sin 2\sqrt{xy}-\sin 2y\right)=0$$

Case I: $\cos 2y=0\Rightarrow 2y=\pm\frac{\pi}{2}+2k\pi\Rightarrow\begin{cases}y=\pm\frac{\pi}{4}+k\pi\Rightarrow\\x=\frac{\pi}{2}-y=\frac{\pi}{2}\mp\frac{\pi}{4}-k\pi\end{cases};k\in\mathbb{Z}$

Case II: $\sin 2\sqrt{xy}-\sin 2y=0\Rightarrow\sin 2\sqrt{xy}=\sin 2y\Rightarrow\sin 2\sqrt{xy}=$

$\sin 2\sqrt{y\cdot y}\Rightarrow\begin{cases}y=0\\x=\frac{\pi}{2}+2k\pi\end{cases};k\in\mathbb{N}$, *because* $x\neq y$; $\begin{matrix}\sin x>0\\y>0\end{matrix}$

1.10 Find $x,y,z>0$ such that:

$$(\mathbf{1}+\boldsymbol{x})^{4}(\mathbf{1}+\boldsymbol{y})^{3}(\mathbf{1}+\boldsymbol{z})^{2}=\mathbf{256}\boldsymbol{xyz}$$

Daniel Sitaru

Solution

$$\frac{x}{1+x}+\frac{y}{(1+x)(1+y)}+\frac{z}{(1+x)(1+y)(1+z)}+\frac{1}{(1+x)(1+y)(1+z)}=$$

$$=\frac{1+x-1}{1+x}+\frac{1+y-1}{(1+x)(1+y)}+\frac{1+z-1}{(1+x)(1+y)(1+z)}$$

$$+\frac{1}{(1+x)(1+y)(1+z)}=$$

$$=\left(1-\frac{1}{1+x}\right)+\left(\frac{1}{1+x}-\frac{1}{(1+x)(1+y)}\right)$$

$$+\left(\frac{1}{(1+x)(1+y)}-\frac{1}{(1+x)(1+y)(1+z)}\right)+$$

$$+\frac{1}{(1+x)(1+y)(1+z)}=1$$

$$1=\frac{x}{1+x}+\frac{y}{(1+x)(1+y)}+\frac{z}{(1+x)(1+y)(1+z)}$$

$$+\frac{1}{(1+x)(1+y)(1+z)}\geq$$

$$\overset{AM-GM}{\geq}4\sqrt[4]{\frac{x}{1+x}\cdot\frac{y}{(1+x)(1+y)}\cdot\frac{z}{(1+x)(1+y)(1+z)}\cdot\frac{1}{(1+x)(1+y)(1+z)}}=$$

$$=4\sqrt[4]{\frac{xyz}{(1+x)^4(1+y)^3(1+z)^2}}$$

$$\frac{1}{4^4}\geq\frac{xyz}{(1+x)^4(1+y)^3(1+z)^2}$$

$$(1+x)^4(1+y)^3(1+z)^2\geq 256xyz$$

Equality holds in AM-GM if:

$$\frac{x}{1+x}=\frac{y}{(1+x)(1+y)}=\frac{z}{(1+x)(1+y)(1+z)}=\frac{1}{(1+x)(1+y)(1+z)}$$

$$x=\frac{y}{1+y};y=\frac{z}{1+z};z=1\Rightarrow y=\frac{1}{2};x=\frac{\frac{1}{2}}{\frac{3}{2}}=\frac{1}{3}$$

Solution: $x=\frac{1}{3};y=\frac{1}{2};z=1$

1.11 Solve for real numbers:

$$\left(\log\left(\frac{x}{x^2+1}\right)+x\right)^3=$$

$$=\left(\log\left(\frac{x}{x^2+1}\right)-x\right)^3+(x-\log(x^3+x))^3+(x+\log(x^3+x))^3$$

Daniel Sitaru

Solution

Denote $a=\log\left(\frac{x}{x^2+1}\right);b=\log(x^3+x)$

$$(a+x)^3=(a-x)^3+(x-b)^3+(x+b)^3$$

$$a^3+3a^2x+3ax^2+x^3=a^3-3a^2x+3ax^2-x^3+$$

$$+x^3-3x^2b+3xb^2-b^3+x^3+3x^2b+3xb^2+b^3$$

$$6a^2x-6xb^2=0,\qquad 6x(a^2-b^2)=0$$

$$a^2-b^2=0\Rightarrow\left(\log\frac{x}{x^2+1}\right)^2-(\log(x^3+x))^2=0$$

$$\left(\log\frac{x}{x^2+1}-\log(x^3+x)\right)\left(\log\frac{x}{x^2+1}+\log(x^3+x)\right)=0$$

$$\log\frac{x}{x^2+1}=\log(x^3+x)\Rightarrow\frac{x}{x^2+1}=x^3+x\Rightarrow$$

$\Rightarrow 1=(x^2+1)^2\Rightarrow x^4+2x^2=0$. *No solution.*

$$\log\frac{x}{x^2+1}=-\log(x^3+x),\qquad \frac{x}{x^2+1}=\frac{1}{x(x^2+1)}\Rightarrow x^2=1\Rightarrow x=1$$

1.12 Find $x,y>0$ such that:

$$\mathbf{(1+x)^3(1+y)^2=27xy}$$

Daniel Sitaru

Solution

$$\frac{x}{1+x}+\frac{y}{(1+x)(1+y)}+\frac{1}{(1+x)(1+y)}=$$

$$=\frac{1+x-1}{1+x}+\frac{1+y-1}{(1+x)(1+y)}+\frac{1}{(1+x)(1+y)}=$$

$$=\left(1-\frac{1}{1+x}\right)+\left(\frac{1}{1+x}-\frac{1}{(1+x)(1+y)}\right)+\frac{1}{(1+x)(1+y)}=1$$

$$1=\frac{x}{1+x}+\frac{y}{(1+x)(1+y)}+\frac{1}{(1+x)(1+y)}\geq$$

$$\overset{AM-GM}{\geq}3\cdot\sqrt[3]{\frac{x}{1+x}\cdot\frac{y}{(1+x)(1+y)}\cdot\frac{1}{(1+x)(1+y)}}=3\sqrt[3]{\frac{xy}{(1+x)^3(1+y)^2}}$$

$$1\geq 3^3\cdot\frac{xy}{(1+x)^3(1+y)^2},\qquad (1+x)^3(1+y)^2\geq 27xy$$

Equality holds in AM-GM if

$$\frac{x}{1+x}=\frac{y}{(1+x)(1+y)}=\frac{1}{(1+x)(1+y)}\Rightarrow y=1$$

$$\frac{x}{1+x}=\frac{1}{(1+x)\cdot 2}\Rightarrow x=\frac{1}{2}$$

1.13 Solve for real numbers:

$$\boldsymbol{x+9^{\log_x 27}+x\cdot 9^{\log_x 27}=279}$$

Marian Ursărescu

Solution (Florentin Vişescu)

$$x\in(0,1)\cup(1,\infty)$$

If $x\in(0,1)\Rightarrow \log_x 27<0\Rightarrow 0<\log_x 27<1; 0<x<1\Rightarrow$

$x+9^{\log_x 27}+x\cdot 9^{\log_x 27}\in(0,3)\Rightarrow$*not solution.*

If $x\in(1,\infty)$. Let $t=\log_x 27>0\Rightarrow \frac{1}{\log_{27}x}=\frac{3}{\log_3 x}=t\Rightarrow$

$$\log_3 x=\frac{3}{t}\Rightarrow x=3^{\frac{3}{t}}.$$

The equation becomes: $3^{\frac{3}{t}}+3^{2t}+3^{\frac{3}{t}+2t}=279.$

Let be the function $f:(0,\infty)\to\mathbb{R}, f(t)=3^{\frac{3}{t}}+3^{2t}+3^{\frac{3}{t}+2t}$

$$f'(t)=\left(-\frac{3}{t^2}\cdot 3^{\frac{3}{t}}+2\cdot 3^{2t}+\left(2-\frac{3}{t^2}\right)3^{\frac{3}{t}+2t}\right)log3$$

$$f"(t)=\left(\frac{9}{t^4}\cdot 3^{\frac{3}{t}}log3+\frac{6}{t^3}\cdot 3^{\frac{3}{t}}+4\cdot 3^{2t}log3+\left(2-\frac{3}{t^2}\right)^2 3^{\frac{3}{t}+2t}log3+\frac{6}{t^3}\cdot\right.$$

$$\left.3^{\frac{3}{t}+2t}\right)log3>0\Rightarrow f\ \text{–convexe}\Rightarrow t\in\left\{1;\frac{3}{2}\right\}\Rightarrow x\in\{9;27\}$$

1.14 Find all $x,y,z>0$ such that:

$$\boldsymbol{4sinx\cdot siny\cdot sinz\cdot sin(x+y+z)=1}$$

Daniel Sitaru

Solution (Marian Dincă)

$$4sinx\cdot siny\cdot sinz\cdot sin(x+y+z)\le 1$$

$$2sinx\cdot siny=cos(x-y)-cos(x+y)$$

$$2sinz\cdot sin(x+y+z)=cos(x+y)-cos(2z+x+y)$$

$$[cos(x-y)-cos(x+y)][cos(x+y)-cos(2z+x+y)]=$$

$$=-cos^2(x+y)+cos(x+y)[cos(x+y)-cos(2z+x+y)]$$

$$-cos(x-y)cos(2z+x+y)-1\le 0$$

$$cos^2(x+y)-cos(x+y)[cos(x+y)-cos(2z+x+y)]$$

$$+cos(x-y)cos(2z+x+y)+1\ge 0$$

Let: $cos(x+y)=t$

$$t^2-t[cos(x+y)-cos(2z+x+y)]+cos(x-y)cos(2z+x+y)+1\ge 0$$

$$\left[t-\frac{1}{2}\left(cos(x-y)+cos(2z+x+y)\right)\right]^2-$$

$$-\frac{1}{4}[cos(x-y)+cos(2z+x+y)]^2+cos(x-y)cos(2z+x+y)+1\ge 0$$

$$\left[t-\frac{1}{2}\left(cos(x-y)+cos(2z+x+y)\right)\right]^2-$$

$$+\frac{4-[cos(x-y)-cos(2z+x+y)]^2}{4}\ge 0$$

But: $cos(x-y)-cos(2z+x+y)=2sin(z+y)sin(z+x)$

So,

$$4-[cos(x-y)-cos(2z+x+y)]^2=4-4sin^2(z+y)sin^2(z+x)\ge 0$$

because: $0\le sin^2(z+y)\le 1; 0\le sin^2(z+x)\le 1$

So, the inequality

$4sinx\cdot siny\cdot sinz\cdot sin(x+y+z)\le 1$ *is true for any* $x,y,z\in\mathbb{R}$.

Equality holds when: $sin(z+y)=\pm 1; sin(z+x)=\pm 1$ *result*:

$$cos(x+z)=0; cos(y+z)=0; t-\frac{1}{2}[cos(x-y)+cos(2z+x+y)]=0$$

Or

$$2cos(x+y)=cos(x-y)+cos(2z+x+y)=2cos(x+z)cos(y+z)$$

But: $cos(x+z)=0; cos(y+z)=0\Rightarrow cos(x+y)=0$

So, $x+z=(2a+1)\frac{\pi}{2}; y+z=(2b+1)\frac{\pi}{2}; x+y=(2c+1)\frac{\pi}{2}$

$$x = (2a - 2b + 2c + 1)\frac{\pi}{4}; y = (-2a + 2b + 2c + 1)\frac{\pi}{4}$$

$$z = (2a + 2b - 2c + 1)\frac{\pi}{4}; a, b, c \in \mathbb{Z}$$

1.15 Find $x, y, z > 0$ such that:

$$\frac{(1+x^2)(1+y^2)}{(1+x)(1+y)} + \frac{(1+y^2)(1+z^2)}{(1+y)(1+z)} + \frac{(1+z^2)(1+x^2)}{(1+z)(1+x)} + 24\sqrt{2} = 36$$

Daniel Sitaru

Solution (Remus Florin Stanca)

Let's prove that: $\sum \frac{(1+x^2)(1+y^2)}{(1+x)(1+y)} = 36 - 24\sqrt{2}$, *let*

$$x = tan\frac{A}{2}, y = tan\frac{B}{2}, z = tan\frac{C}{2}, A, B, C \in \left[0, \frac{\pi}{2}\right],$$

$$sinA = \frac{2x}{x^2+1}, sinB = \frac{2y}{y^2+1}, sinC = \frac{2z}{z^2+1}$$

$$sinA + \frac{2}{x^2+1} = 2 \cdot \frac{x+1}{x^2+1} \Rightarrow sinA + 2cos^2\frac{A}{2} - 1 + 1 = 2 \cdot \frac{x+1}{x^2+1} \Rightarrow$$

$$sinA + cosA + 1 = 2 \cdot \frac{x+1}{x^2+1} \Rightarrow \frac{x^2+1}{x+1} = \frac{2}{sinA + cosA + 1} \Rightarrow$$

$$\sum_{cyc} \frac{(1+x^2)(1+y^2)}{(1+x)(1+y)} = \sum_{cyc} \frac{4}{(sinA + cosA + 1)(sinB + cosB + 1)}$$

$$\geq 36 - 24\sqrt{2} \Leftrightarrow$$

$$\sum_{cyc} \frac{1}{(sinA + cosA + 1)(sinB + cosB + 1)} \geq 9 - 6\sqrt{2}; (1)$$

$$\sum_{cyc} \frac{1}{(sinA + cosA + 1)(sinB + cosB + 1)} \overset{Bergstrom}{\geq} \frac{9}{\sum_{cyc}(sinA + cosA + 1)(sinB + cosB + 1)}; (2)$$

Let's prove that:

$$\frac{9}{\sum_{cyc}(sinA + cosA + 1)(sinB + cosB + 1)} \geq 9 - 6\sqrt{2}$$

We need to prove that:

$$\frac{3}{\sum_{cyc}(sinA+cosA+1)(sinB+cosB+1)} \geq 3-2\sqrt{2}=\frac{1}{\left(\sqrt{2}+1\right)^2} \Leftrightarrow$$

$$\sum_{cyc}(sinA+cosA+1)(sinB+cosB+1) \leq 9+6\sqrt{2} \Leftrightarrow$$

$$\sum_{cyc}(sinAsinB+sinAcosB+sinA+sinAsinB+cosAcosB+cosA+sinB$$
$$+cosB+1) \leq 9+6\sqrt{2} \Leftrightarrow$$

$$\sum_{cyc}\left(cos(A-B)+sin(A+B)\right)+2(sinA+sinB+sinC)$$
$$+2(cosA+cosB+cosC) \leq 6+6\sqrt{2} \Leftrightarrow$$

$$\sum_{cyc}\left(cos(A-B)+cos\left(\frac{\pi}{2}-A-B\right)\right)+2(sinA+sinB+sinC)$$
$$+2(cosA+cosB+cosC) \leq 6+6\sqrt{2} \Leftrightarrow$$

$$2\sum_{cyc}(cos\left(\frac{\pi}{4}-B\right)cos\left(\frac{\pi}{4}-A\right)+2\sqrt{2}\sum_{cyc}cos\left(\frac{\pi}{4}-A\right) \leq 6+6\sqrt{2}$$

Let: $cos\left(\frac{\pi}{4}-A\right)=x_1;\ cos\left(\frac{\pi}{4}-B\right)=x_2;\ cos\left(\frac{\pi}{4}-C\right)=x_3 \Leftrightarrow$

$$(x_1x_2+x_2x_3+x_3x_1)+\sqrt{2}(x_1+x_2+x_3) \leq 3+3\sqrt{2};\ (3)$$

$$\frac{(x_1+x_2+x_3)^2}{3} \geq x_1x_2+x_2x_3+x_3x_1 \Rightarrow$$

$$(x_1x_2+x_2x_3+x_3x_1)+\sqrt{2}(x_1+x_2+x_3) \leq \frac{(x_1+x_2+x_3)^2}{3}+\sqrt{2}(x_1+x_2+x_3) \stackrel{(3)}{\Rightarrow}$$

We need to prove that:

$\frac{(x_1+x_2+x_3)^2}{3}+\sqrt{2}(x_1+x_2+x_3) \leq 3+3\sqrt{2}$ let $x_1+x_2+x_3=s$ *and we know that* $0 \leq s \leq 3 \Rightarrow s^2+3s\sqrt{2}-9-9\sqrt{2} \leq 0 \stackrel{(1),(2)}{\Longrightarrow}$ *we peove that (1),(2) are true.*

But we have in the equality case $s^2+3s\sqrt{2}-9-9\sqrt{2}=0;\ \Delta=54+36\sqrt{2} \Rightarrow s=3,$ *because* $s \geq 0 \Rightarrow$

$$\sum_{cyc}cos\left(\frac{\pi}{4}-A\right)=3 \Rightarrow 2\sum_{cyc}sin^2\left(\frac{\frac{\pi}{4}-A}{2}\right)=0 \Rightarrow A=B=C=\frac{\pi}{4} \Rightarrow$$

$$x=y=z=tan\frac{\pi}{8}$$

$$tan\frac{\pi}{8}=\frac{sin\frac{\pi}{8}}{cos\frac{\pi}{8}}=\frac{\sqrt{\frac{1-cos\frac{\pi}{4}}{2}}}{\sqrt{\frac{1+cos\frac{\pi}{4}}{2}}}=\sqrt{\frac{2-\sqrt{2}}{2+\sqrt{2}}}=\sqrt{2}-1\Rightarrow x=y=z=\sqrt{2}-1.$$

1.16 Let be the equation:

$$\boldsymbol{x^9+\frac{15}{8}x^6+\frac{75}{64}x^3-x+\frac{445}{512}=0}$$

If S −sum of all real roots then find $[S]$, $[*]$ −GIF

Rajeev Rastogi

Solution (Khanh Hung Vu)

$$x^9+\frac{15}{8}x^6+\frac{75}{64}x^3-x+\frac{445}{512}=0;(1)$$

Put: $t=x^3;(2)$, *we have the given equation equivalent to:*

$$t^3+\frac{15}{8}t^2+\frac{75}{64}t-x+\frac{445}{512}=0\Rightarrow x=t^3+\frac{15}{8}t^2+\frac{75}{64}t+\frac{445}{512}$$

$$=\left(t+\frac{5}{8}\right)^3+\frac{5}{8};(3)$$

From (2),(3), we have $x^3+x=t+\left(t+\frac{5}{8}\right)^3+\frac{5}{8};(4)$

Put $f(t)=t^3+t$, *we have* $f'(t)=3t^2+1>0$ *so* $f(t)$ −*is a increasing function.*

So, from (4) we get $x=t+\frac{5}{8}\Rightarrow x=x^3+\frac{5}{8}\Rightarrow x^3-x+\frac{5}{8}=0\Rightarrow 8x^3-8x+5=0.$

Put $g(x)=8x^3-8x+5, g'(x)=24x^2-8.$

So, $g'(x)=0\Leftrightarrow x=\pm\frac{\sqrt{3}}{3}$. *We have* $g\left(-\frac{\sqrt{3}}{3}\right)=5+\frac{16\sqrt{3}}{9}$ *and* $\left(\frac{\sqrt{3}}{3}\right)=5-\frac{16\sqrt{3}}{9}$

So, $8x^3-8x+5=0$ *has only one root in* $\left(-\infty,-\frac{\sqrt{3}}{3}\right)$.

On the other hand, we have $f(-2)f(-1) < 0$ *then the equation (1) has only one root in* $(-2,-1)$. *So,* $[S] = -2$.

1.17 Solve for integers:

$$\boldsymbol{y^3 = x^2 + 2x}$$

Jalil Hajimir

Solution(Bedri Jajrizi)

Let: $x + 1 = k \Rightarrow y^3 = (k-1)(k+1);\ (k-1, k+1) \in \{1,2\}$

Case i) $(k-1, k+1) = 1 \Rightarrow k$ *−even.*

$(k-1, k+1) = cube \Rightarrow \exists a, b \in \mathbb{Z}$ *such that* $k + 1 = a^3, k - 1 = b^3 \Rightarrow a^3 - b^3 = 2$.

It's clear that: $a^3 = 1, b^3 = -1 \Rightarrow a = 1, b = -1$

So: $k + 1 = 1 \Rightarrow k = 0, y = -1$. *Solutions is:* $x = 1, y = -1$

Case ii) $(k-1, k+1) = 2 \Rightarrow k$ *is odd,* y *is even.*

It's clear that $k \equiv 1{,}3(mod4)$.

Also (x, y) *is solution*$\Leftrightarrow$ $(-k, y)$ *is solution.*

So assume that $k \equiv 1(mod4) \Rightarrow k + 1 \equiv 2(mod4), k - 1 \equiv 0(mod4)$

We can write the equation in the form:

$$\left(\frac{y}{2}\right)^3 = \frac{k+1}{2} \cdot \frac{k-1}{4} \text{ where } \left(\frac{k+1}{2}, \frac{k-1}{4}\right) = 1 \text{ because } (k+1, k-1) = 1$$

Let: $\frac{k+1}{2} = a^3, \frac{k-1}{2} = b^3 \Rightarrow 2a^3 - 1 = k = 4b^3 + 1 \Rightarrow a^3 - 2b^3 = 1$

$(a, b) \in \{(0,1), (-1,1)\} \Rightarrow k \in \{-1, -3\} \Rightarrow x \in \{-2, -4\}$

Solutions are: $(x, y) \in \{(-2,0), (-4,2), (0,0), (2,2)\}$

1.18

$$\boldsymbol{f(x) = x^8 - 9x^7 + 31x^6 + a_0x^5 + a_1x^4 + a_2x^3 + a_3x^2 + a_4x + a_5 \in \mathbb{R}[x]}$$

If $f(x)$ has all roots real prove that its lie in $[-2,4]$.

Rajeev Rastogi

Solution(Khanh Hung Vu)

Support $f(x)$ has all roots which is $x_i, i = \overline{1,8}$

By Viete theorem for polynomial $f(x)$, we have:

$$\begin{cases} \sum_{i=1}^{8} x_i = 9; (1) \\ \sum_{\substack{1,j\in[1,8] \\ i<j}} x_i x_j = 31; (2) \end{cases}$$

We have (2) equivalent to:

$$\left(\sum_{i=1}^{8} x_i\right)^2 - \sum_{i=1}^{8} x_i^2 = 2\cdot 31 \overset{(1)}{\Rightarrow} \sum_{i=1}^{8} x_i^2 - 62 = 19$$

On the other hand, by CBS inequality, we have:

$$\sum_{i=1}^{8} x_i^2 \geq \frac{1}{7}\left(\sum_{i=1}^{8} x_i\right)^2 \text{ and since } \sum_{i=2}^{8} x_i = 9 - x_1 \Rightarrow \sum_{i=1}^{8} x_i^2 \geq x_1^2 + \frac{(9-x_1)^2}{7}$$

$$x_1^2 + \frac{(9-x_1)^2}{7} \leq 19 \Rightarrow x_1 \in \left[\frac{9-\sqrt{487}}{8}, \frac{9+\sqrt{487}}{8}\right] \text{ or } x_1 \in [-2,4]$$

Similarly for, $x_2, x_3, \dots, x_8 \in [-2,4]$. So, $x_i \in [-2,4], i = \overline{1,8}$

1.19 Find all real roots:

$$\boldsymbol{64x^5(x-1) + 32x^2(x^2+x+1) - 64x + 19 = 0}$$

Daniel Sitaru

Solution(Abner Chinga Bazo)

$$64x^5(x-1)+32x^2(x^2+x+1)-64x+19=0$$

$$64x^6-64x^5+32x^4+32x^3+32x^2-64x+19=0\,;(Horner)\Leftrightarrow$$

$$(2x-1)^2(16x^4+4x^2+12x+19)=0$$

$$16x^4+(4x^2+12x+9)+10=0$$

$$16x^4+(2x+3)^2+10>0,\forall x\in\mathbb{R}\Rightarrow 2x-1=0\Rightarrow x=\frac{1}{2}$$

So, $S=\left\{\frac{1}{2}\right\}$

1.20 Solve for real numbers:

$$|\boldsymbol{cosx}|+|\boldsymbol{cosy}|=\sqrt{(\boldsymbol{2}+\boldsymbol{sinx}+\boldsymbol{siny})(\boldsymbol{2}-\boldsymbol{sinx}-\boldsymbol{siny})}$$

Daniel Sitaru

Solution(Ravi Prakash)

$$|cosx|+|cosy|=\sqrt{(2+sinx+siny)(2-sinx-siny)};\quad(1)$$

$$\Rightarrow(|cosx|+|cosy|)^2=4-(sinx+siny)^2\Rightarrow$$

$$cos^2x+cos^2y+2|cosx||cosy|=4-2sinxsiny-sin^2x-sin^2y\Leftrightarrow$$

$$|cosxcosy|+sinxsiny=1$$

If $cosxcosy\geq 0$ we get:

$$cos(x-y)=1\Leftrightarrow x-y=2r\pi\Leftrightarrow x=2r\pi+y,r\in\mathbb{Z}\Rightarrow$$

$$y=2n\pi+x,n\in\mathbb{Z}$$

If $cosxcosy\leq 0$ we get:

$$cosxcosy+sinxsiny=1\Leftrightarrow cos(x+y)=-1\Leftrightarrow x+y=(2r+1)\pi\Leftrightarrow$$
$$y=(2r+1)\pi-x,r\in\mathbb{Z}$$

Thus, $y=2n\pi+x,n\in\mathbb{Z}$ or $y=(2r+1)\pi-x,r\in\mathbb{Z}$

1.21

$$A=\left\{x \,\middle|\, x\in\mathbb{Z},\left[\frac{x^7-15x^5+49x^3-36x}{56}\right]=0,[*]-GIF\right\}$$

Find:

$$\Omega=\sum_{x\in A}x$$

Daniel Sitaru

Solution(Djeeraj Badera)

Given that:

$$x\in\mathbb{Z},\left[\frac{x^7-15x^5+49x^3-36x}{56}\right]=0;\quad (A)$$

From def. of GIF:

$$0\le \underbrace{x^7-15x^5+49x^3-36x}_{f(x)}<56;\quad (B)$$

(for satisfying condition A)

Case 1.

$$x^7-15x^5+49x^3-36x=0\Leftrightarrow x(x^6-15x^4+49x^2-36)=0\Leftrightarrow$$
$$x(x^2-1)(x^2-9)(x^2-4)=0\Leftrightarrow x\in\{0,\pm1,\pm2,\pm3\}$$

Case 2.

$$x\in\mathbb{Z}\ for\ x^7-15x^5+49x^3-36x<56$$
$$x\in\mathbb{Z};\ x\ge 4$$
$$f(x)=x(x^2-1)(x^2-9)(x^2-4)$$
$$f'(x)=(x^2-1)(x^2-9)(x^2-4)+2x^2(x^2-9)(x^2-4)+2x^2(x^2-1)(x^2-4)++2x^2(x^2-1)(x^2-9)$$

For all $x\in\mathbb{Z}$ where $x\ge4\Rightarrow f'(x)>0\Rightarrow f-$increasing.

$f(4)=4(16-1)(16-9)(16-4)>56\Rightarrow x\ge4$ not satisfy condition (B) similarly for $x\le-4$.

Possible value of x is $\{0,\pm1,\pm2,\pm3\}$

$$\Omega=\sum_{x\in A}x=0+1-1+2-2+3-3=0$$

FUNCTIONAL EQUATIONS

2.1 Find all functions $f:(0,+\infty)\to\mathbb{R}$ such that:

$$f(xy)\le xf(x)+yf(y)\le \log(xy),\forall x,y>0$$

Marian Ursărescu

Solution

$$x=y=1\Rightarrow f(1)\le 2f(1)\le 0\Rightarrow f(1)\le 0 \text{ but } f(1)\ge 0\Rightarrow f(1)=0$$

$$y=1\Rightarrow f(x)\le xf(x)\le \log x\Rightarrow f(x)\le \frac{\log x}{x};\ \ (1)$$

$$y=\frac{1}{x}\Rightarrow f(1)\le xf(x)+\frac{1}{x}f\left(\frac{1}{x}\right)\le \log 1\Rightarrow 0\le xf(x)+\frac{1}{x}f\left(\frac{1}{x}\right)\le 0\Rightarrow$$

$$xf(x)+\frac{1}{x}f\left(\frac{1}{x}\right)=0\Rightarrow f\left(\frac{1}{x}\right)=-x^2f(x);\ \ (2)$$

$$\text{In } (1)x\to\frac{1}{x}\Rightarrow f\left(\frac{1}{x}\right)\le\frac{\log\left(\frac{1}{x}\right)}{\frac{1}{x}}\Rightarrow f\left(\frac{1}{x}\right)\le -x\log x;\ \ (3)$$

$$\text{From } (1),(2)\Rightarrow -x^2f(x)\le -x\log x\Rightarrow x^2f(x)\ge x\log x\Rightarrow f(x)\ge \frac{\log x}{x};\ \ (4)$$

$$\text{From } (3),(4)\Rightarrow f(x)=\frac{\log x}{x}$$

2.2 $$\begin{cases} f,g,h:\mathbb{R}\to\mathbb{R}\\ f(x)+g(x)+h(x)=3x+3,\forall x\in\mathbb{R}\\ f^2(x)+g^2(x)+h^2(x)=3x^2+6x+5,\forall x\in\mathbb{R}\\ f^3(x)+g^3(x)+h^3(x)=3x^3+9x^2+15x+9,\forall x\in\mathbb{R}\end{cases}$$

Solve for real numbers: $f(x)\cdot g(x)\cdot h(x)=0$

Daniel Sitaru

Solution (Adrian Popa)

$$\textit{Denote}:\begin{cases} f(x)+g(x)+h(x)=S_1\\ f(x)\cdot g(x)+g(x)\cdot h(x)+h(x)\cdot f(x)=S_2\\ f(x)\cdot g(x)\cdot h(x)=S_3\end{cases}$$

$$S_1 = 3x + 3;\ S_1^2 - 2S_2 = 3x^2 + 6x + 5 \Rightarrow 9x^2 + 18x + 9 - 2S_2 = 3x^2 + 6x + 5$$

$$\Rightarrow 2S_2 = 6x^2 + 12x + 4 \Rightarrow S_2 = 3x^2 + 6x + 2$$

$$S_1^3 - 3S_1S_2 + 3S_3 = 3x^3 + 9x^2 + 15x + 9$$

$$27(x^3 + 3x^2 + 3x + 1) - 9(x + 1)(3x^2 + 6x + 2) + 3S_3$$

$$= 3x^3 + 9x^2 + 15x + 9$$

$$3S_3 = 3x^3 + 9x^2 + 6x \Rightarrow S_3 = x^3 + 3x^2 + 2x$$

$$f(x) \cdot g(x) \cdot h(x) = x^3 + 3x^2 + 2x = 0 \Leftrightarrow$$

$$x(x^2 + 3x + 2) = 0 \Rightarrow x \in \{-2; -1; 0\}$$

2.3 Find the function $f: \mathbb{R} \to \mathbb{R}$ continuous such that:

$$\boldsymbol{f(x) + f(x^2) = 2, \forall x \in \mathbb{R}.}$$

Sridhar Rao

Solution (Ravi Prakash)

$$f(x) + f(x^2) = 2, \forall x \in \mathbb{R}; (1)$$

Replace x *by* $-x$ *we get:* $f(-x) + f((-x)^2) = 2 \Rightarrow$

$f(-x) + f(x^2) = 2; \forall x \in \mathbb{R}; (2)$ *From (1),(2) we get:* $f(-x) = f(x), \forall x \in \mathbb{R}$

Also, from (1), $2f(0) = 2 \Rightarrow f(0) = 1,\ 2f(1) = 2 \Rightarrow f(1) = 1$

Let $0 < x < 1$. *From (1) we have:*

$$f(x) = 2 - f(x^2) = 2 - [2 - f((x^2)^2)] = f(x^4); (3) \Rightarrow$$

$$f(x) = f(x^4) = f(x^{16}) = \dots = f\left(x^{4^n}\right), \forall n \in \mathbb{N}$$

As $0 < x < 1, x^{4^n} \xrightarrow[n\to\infty]{} 0$

Since f *–is continuous on* $\mathbb{R} \Rightarrow \lim_{n\to\infty} f\left(x^{4^n}\right) = f(0) = 1 \Rightarrow$

$f(x) = 1, \forall x \in (0,1)$.*Let* $x > 1$*, then from (3)*

$$f(x) = f\left(x^{\frac{1}{4}}\right) = f\left(x^{\frac{1}{16}}\right) = \dots = f\left(x^{\frac{1}{4^n}}\right), \forall n \in \mathbb{N}$$

As $x > 1, x^{\frac{1}{4^n}} \xrightarrow[n\to\infty]{} 1, f$ *–continuous on* $\mathbb{R} \Rightarrow \lim_{n\to\infty} f\left(x^{\frac{1}{4^n}}\right) = f(1) = 1 \Rightarrow$

$f(x) = 1, \forall x > 1$. *Thus,* $f(x) = 1, \forall x \in \mathbb{R}$.

SYSTEMS

3.1 Solve for real numbers:

$$\begin{cases} 0 < x, y < \frac{\pi}{2}, x + y = \frac{5\pi}{6} \\ 4\sin^2 x \sin^2 y(\sin^2 x + \sin^2 y) + 4\cos^2 x \cos^2 y(\cos^2 x + \cos^2 y) = \sin^2 2x + \sin^2 2y \end{cases}$$

Daniel Sitaru

Solution(Khanh Hung Vu)

$$\begin{cases} 0 < x, y < \frac{\pi}{2}, x + y = \frac{5\pi}{6}; (1) \\ 4\sin^2 x \sin^2 y(\sin^2 x + \sin^2 y) + 4\cos^2 x \cos^2 y(\cos^2 x + \cos^2 y) = \sin^2 2x + \sin^2 2y; (2) \end{cases}$$

Let: $\begin{cases} \sin^2 x = a \\ \sin^2 y = b \end{cases} \Rightarrow \begin{cases} \cos^2 x = 1 - a \\ \cos^2 y = 1 - b \end{cases} \Rightarrow \begin{cases} \sin^2 2x = 4\sin^2 x\cos^2 x = 4a(1-a) \\ \sin^2 2y = 4\sin^2 y\cos^2 y = 4b(1-b) \end{cases}$

$$(2) \Leftrightarrow 4ab(a+b) + 4(1-a)(1-b)(2-a-b) = 4a(1-a) + 4b(1-b)$$

$$\Leftrightarrow ab(a+b) + (1-a)(1-b)(2-a-b) = a(1-a) + b(1-b)$$

$$2(a^2 + b^2 + 2ab - 2a - 2b + 1) = 0 \Rightarrow 2(a+b-1)^2 = 0 \Rightarrow a + b = 1$$
$$\Rightarrow a = 1 - b \Rightarrow \sin^2 x = \cos^2 y$$

So, we have 2 cases:

1) $\sin x = \cos y \Rightarrow \sin x = \sin\left(\frac{\pi}{2} - y\right) \Rightarrow \begin{cases} x = \frac{\pi}{2} - y + 2k\pi \\ x = \pi - \left(\frac{\pi}{2} - y\right) + 2k\pi \end{cases} \Rightarrow$

$$\begin{cases} x + y = \frac{\pi}{2} + 2k\pi \\ x - y = \frac{\pi}{2} + 2k\pi \end{cases}; k \in \mathbb{Z}$$

Since $x + y = \frac{5\pi}{6}$ so $\nexists k \in \mathbb{Z}: x + y = \frac{\pi}{2} + 2k\pi$

$$\left(since \frac{5\pi}{6} = \frac{\pi}{2} + 2k\pi \Rightarrow k = \frac{1}{6} \notin \mathbb{Z}\right)$$

So we have:

$$\begin{cases} 0 < x, y < \frac{\pi}{2}, k \in \mathbb{Z} \\ x + y = \frac{5\pi}{6} \\ x - y = \frac{\pi}{2} + 2k\pi \end{cases} \Rightarrow \begin{cases} 0 < x, y < \frac{\pi}{2}, k \in \mathbb{Z} \\ x = \frac{2\pi}{3} + k\pi \\ y = \frac{\pi}{6} - k\pi \end{cases} \Rightarrow \text{No root.}$$

2) $sinx = -cosy \Rightarrow sinx = sin\left(\frac{\pi}{2} - y\right) \Rightarrow \begin{cases} x = y - \frac{\pi}{2} + 2k\pi \\ x = \pi - \left(y - \frac{\pi}{2}\right) + 2k\pi \end{cases} \Rightarrow$

$$\begin{cases} x - y = -\frac{\pi}{2} + 2k\pi \\ x + y = \frac{3\pi}{2} + 2k\pi \end{cases}, k \in \mathbb{Z}$$

Since $x + y = \frac{5\pi}{6}$ so $\nexists k \in \mathbb{Z}: x + y = \frac{3\pi}{2} + 2k\pi$

$$\left(since \frac{5\pi}{6} = \frac{3\pi}{2} + 2k\pi \Rightarrow k = -\frac{1}{3} \notin \mathbb{Z}\right)$$

So we have:

$$\begin{cases} 0 < x, y < \frac{\pi}{2}, k \in \mathbb{Z} \\ x + y = \frac{5\pi}{6} \\ x - y = -\frac{\pi}{2} + 2k\pi \end{cases} \Rightarrow \begin{cases} 0 < x, y < \frac{\pi}{2}, k \in \mathbb{Z} \\ x + y = \frac{\pi}{6} + k\pi \\ y = \frac{2\pi}{3} - k\pi \end{cases} \Rightarrow \text{No root}$$

So the system has no root.

3.2 Solve the system of equations:

$$\begin{cases} 2\left(\frac{x^3}{y^2} + \frac{y^3}{x^2}\right) = \sqrt[4]{8(x^4 + y^4)} + 2\sqrt{xy} \\ 16x^5 - 20x^3 + 5\sqrt{xy} = \sqrt{\frac{y+1}{2}} \end{cases}$$

Hoang Le Nhat Tung

Solution (Soumava Chakraborty)

$$2\left(\frac{x^3}{y^2}+\frac{y^3}{x^2}\right)\overset{(1)}{=}\sqrt[4]{8(x^4+y^4)}+2\sqrt{xy}\ \&\ 16x^5-20x^3+5\sqrt{xy}\overset{(2)}{=}\sqrt{\frac{y+1}{2}}$$

Of course $x,y\neq 0\ \&\ xy>0\Rightarrow x,y<0$ *or* $x,y>0$. *If* $x,y<0$, *then LHS of (1)* <0, *but RHS of (1)* $>0\Rightarrow x,y<0$ *is impossible* $\therefore x,y>0$.

$$\text{Now, } x^4+y^4\le 2(x^2-xy+y^2)^2$$

$$\Leftrightarrow x^4+y^4+6x^2y^2-6xy(x^2+y^2)\ge 0$$

$$\Leftrightarrow (x^2+y^2)^2+4x^2y^2-4xy(x^2+y^2)\ge 0\Leftrightarrow$$

$$\Leftrightarrow (x^2+y^2-2xy)^2\ge 0\to \text{true}$$

$$\therefore \sqrt{x^4+y^4}\le\sqrt{2}|x^2-xy+y^2|=\sqrt{2}(x^2-xy+y^2)$$

$$\left(\because x^2-xy+y^2=\frac{1}{4}(x+y)^2+\frac{3}{4}(x-y)^2>0\right)$$

$$\Rightarrow\sqrt[4]{\frac{x^4+y^4}{2}}\le\sqrt{x^2-xy+y^2}\Rightarrow\sqrt[4]{8(x^4+y^4)}\le 2\sqrt{x^2-xy+y^2}\Rightarrow$$

$$\Rightarrow\sqrt[4]{8(x^4+y^4)}+2\sqrt{xy}$$

$$\le 2\left(\sqrt{x^2-xy+y^2}+\sqrt{xy}\right)\overset{CBS}{\le}2\sqrt{2}\sqrt{x^2-xy+y^2+xy}=$$

$$=2\sqrt{2}\sqrt{x^2+y^2}\ \underset{(a)}{\overset{?}{\le}}\ \frac{2(x^2+y^2)^2}{xy(x+y)}\Leftrightarrow\frac{(x^2+y^2)^4}{x^2y^2(x+y)^2}\overset{?}{\ge}2(x^2+y^2)\Leftrightarrow$$

$$\Leftrightarrow (x^2+y^2)^3\ \underset{(b)}{\overset{?}{\ge}}\ 2x^2y^2(x+y)^2$$

$$\text{Now, } (x^2+y^2)^3=(x^2+y^2)(x^2+y^2)^2\overset{Chebyshev}{\ge}\frac{1}{2}(x+y)^2(x^2+y^2)\overset{A-G}{\ge}$$

$\ge\frac{1}{2}(x+y)^2\cdot 4x^2y^2=2x^2y^2(x+y)^2\Rightarrow$ *(b) is true* $\Rightarrow$ *(a) is true* $\Rightarrow$

$\Rightarrow$ *RHS of (1)* $\overset{(i)}{\le}\frac{2(x^2+y^2)^2}{xy(x+y)}$, *equality at* $x=y$.

Again, LHS of (1)$=2\left(\frac{x^4}{xy^2}+\frac{y^4}{x^2y}\right)\underset{(ii)}{\overset{Bergstrom}{\ge}}2\frac{(x^2+y^2)^2}{xy(x+y)}$, *equality at* $x=y$.

(i), (ii) $\Rightarrow$ *LHS of (1) = RHS of (1)* $= \frac{2(x^2+y^2)^2}{xy(x+y)}$ &

$\because$ *respective equalities occur at* $x = y$

Putting $y = x$ *in (2), we get:* $16x^5 - 20x^3 + 5x = \sqrt{\frac{x+1}{2}} \Rightarrow$

$$\Rightarrow 16x^5 - 20x^3 + 5x - 1 = \sqrt{\frac{x+1}{2}} - 1 \Rightarrow$$

$$(x-1)(4x^2+2x-1)^2 = \frac{\frac{x+1}{2}-1}{\sqrt{\frac{x+1}{2}}+1} = \frac{x-1}{2\left(1+\sqrt{\frac{x+1}{2}}\right)}.$$

One possibility is $x = 1 \Rightarrow x = y = 1$ *is a solution when*

$x \neq 1, 2(4x^2+2x-1)^2\left(1+\sqrt{\frac{x+1}{2}}\right) = 1$. *Let* $\sqrt{\frac{x+1}{2}} = t$. *Then, we have:*

$$(2+2t)(4(2t^2-1)^2 + 2(2t^2-1) - 1)^2 - 1 = 0 \Rightarrow$$

$$\Rightarrow (2t+1)(8t^3-6t+1)(32t^5+16t^4-32t^3-12t^2+6t+1) = 0$$

The equations yields two acceptable solutions:

$t = \cos^2\frac{2\pi}{9} \Rightarrow \sqrt{\frac{x+1}{2}} = \cos\frac{2\pi}{9} \Rightarrow x = \cos\frac{4\pi}{9}$ & $t \approx .84125 \Rightarrow x \approx .415415$

$\therefore$ *all possible solutions are*

$$(x = y = 1), \left(x = y = \cos\frac{4\pi}{9}\right), (x = y \approx .415415)$$

3.3 Solve for real numbers:

$$\begin{cases} 2x^2 + y^2 = x\sqrt{y}(2\sqrt{x} + \sqrt{y}) \\ x^5 - 3\sqrt{xy} + 4 \leq \sqrt{2y^2 - 2x + 1} + \sqrt[3]{3x^3 - 3xy + 1} \end{cases}$$

Hoang Le Nhat Tung

Solution (Tran Hong)

$$x, y \geq 0; \textit{let } y = tx \quad (t \geq 0)$$

$$2x^2 + t^2x^2 = x\sqrt{tx}\left(2\sqrt{x} + \sqrt{tx}\right) \Leftrightarrow x^2(2 + t^2) = x^2\sqrt{t}\left(2 + \sqrt{t}\right)$$

$$\Leftrightarrow \left[\begin{array}{c} x^2 = 0 \\ 2 + t^2 - \left(2\sqrt{t} + t\right) = 0 \end{array}\right. \Leftrightarrow \left[\begin{array}{c} x = 0 \\ t^2 - \left(t + 2\sqrt{t}\right) + 2 = 0 \quad (*) \end{array}\right.$$

$$\therefore \textit{If } x = 0 \Rightarrow y = 0 \textit{ then:}$$

$$0^5 - 3\sqrt{0 \cdot 0} + 4 \leq \sqrt{2 \cdot 0^2 - 2 \cdot 0 + 1} + \sqrt[3]{3 \cdot 0^3 - 3 \cdot 0 \cdot 0 + 1}$$

$$\Leftrightarrow 4 \leq 2 \ \textit{(contrary)}. \because t^2 - \left(t + 2\sqrt{t}\right) + 2 = 0$$

$$\textit{Let } u = \sqrt{t}\ ,\ u^4 - (u^2 + 2u) + 2 = 0$$

$$\Leftrightarrow u^4 - u^2 - 2u + 2 = 0 \Leftrightarrow (u - 1)^2[(u + 1)^2 + 1] = 0$$

$$\Leftrightarrow u = 1 \Leftrightarrow t = 1 \Leftrightarrow y = x$$

$$\Rightarrow x^5 - 3x + 4 \leq \sqrt{2x^2 - 2x + 1} + \sqrt[3]{3x^3 - 3x^2 + 1} \quad \textit{(1)}$$

We must show that

$$\sqrt{2x^2 - 2x + 1} + \sqrt[3]{3x^3 - 3x^2 + 1} \leq x^5 - 3x + 4 \quad \textit{(2)}$$

$$x^5 - 3x + 4 \geq 2\sqrt{2x^2 - 2x + 1} \quad \textit{(3)}$$

$$\Leftrightarrow (x^5 - 3x + 4)^2 \geq 4(2x^2 - 2x + 1)$$

$$\Leftrightarrow (x - 1)^2(x^8 + 2x^7 + 3x^6 + 4x^5 - x^4 + 2x^3 + 5x^2 + 8x + 12) \geq 0$$

It is true with $x \geq 0$*:*

$$\because 0 \leq x \leq 1 \Rightarrow x^8 + 2x^7 + 3x^6 + 4x^5 + 2x^3 + 5x^2 + 8x + 12 - x^4$$
$$\geq 12 - 1 = 11 > 0$$

$$\because x > 1 \Rightarrow x^4(x^2 - 1) + 2x^7 + 3x^6 + 4x^5 + 2x^3 + 5x^2 + 8x + 12 > 0$$

$$x^5 - 3x + 4 \geq 2\sqrt[3]{3x^3 - 3x^2 + 1} \quad \textit{(4)}$$

$$\Leftrightarrow (x^5 - 3x + 4)^3 \geq 8(3x^3 - 3x^2 + 1)$$

$$\Leftrightarrow (x - 1)^2(x^{13} + 2x^{12} + 3x^{11} + 4x^{10} - 4x^9 +$$
$$+4x^7 + 8x^6 + 38x^5 - 2x^4 + 5x^3 + 12x^2 - 32x + 56) \geq 0$$

It is true with $x \geq 0$*:*

$$\because 0 \le x \le 1 \Rightarrow x^{13} + 2x^{12} + 3x^{11} + 4x^{10} + 8x^5 + 5x^3 + 12x^2 + 56 -$$

$$-(4x^9 + 2x^4 - 32x) > 56 - (4 + 2 + 32) = 12 > 0$$

$$\because x > 1 \Rightarrow 4x^9(x - 1) > 0; 2x^4(x - 1) > 0$$

$$56 + \{x^{13} + 2x^{12} + 3x^{11} + 6x^5 + 5x^3 + 12x^2 - 32x\} =$$

$$= 56 + x(x^{12} + 2x^{11} + 3x^{10} + 6x^4 + 5x^2 + 12x - 32)$$

$$> 56 + (1 + 2 + 3 + 6 + 5 + 12 - 32) = 53 > 0$$

From (3) and (4) $\Rightarrow$ *(2) true.*

From (1) and (2) we have equality $\Leftrightarrow x = 1 \Rightarrow y = x = 1.$

Hence: $(x, y) = (1,1)$

3.4 Solve for real numbers:

$$\begin{cases} \boldsymbol{x^3 + y^3 = \sqrt{xy}(x^2 + y^2)} \\ \boldsymbol{6\sqrt[3]{2x^2 - 2y + 1} + 4 \cdot \sqrt[4]{3x^2 \cdot y - 2x^4} = 2y^5 - 5\sqrt{xy} + 13} \end{cases}$$

Hoang Le Nhat Tung

Solution (George Florin Șerban)

$$\sqrt{xy} = \frac{x^3 + y^3}{x^2 + y^2} \underset{Mg \le Ma}{\le} \frac{x + y}{2} \Rightarrow 2(x^3 + y^3) \le (x + y)(x^2 + y^2)$$

$$2x^3 + 2y^3 \le x^3 + y^3 + xy(x + y), x^3 + y^3 \le xy(x + y)$$

$$(x + y)(x^2 - xy + y^2) - xy(x + y) \le 0, (x + y)(x - y)^2 \le 0$$

$(x + y), (x - y)^2 \ge 0.$ *If* $x + y \le 0$ *and* $xy \ge 0 \Rightarrow$

$\Rightarrow x, y \le 0 \Rightarrow \sqrt{xy} = \frac{x^3+y^3}{x^2+y^2} \le 0$ *false,* $x^3, y^3 \le 0$

$$\Rightarrow x, y \ge\Rightarrow (x + y)(x - y)^2 \ge 0 \Rightarrow (x + y)(x - y)^2 = 0$$

I. If $x + y = 0, x, y \ge 0 \Rightarrow x = y = 0; 6 \cdot 1 + 4 \cdot 0 = 0 - 0 + 13$ *false.*

II. If $x = y \Rightarrow 6\sqrt[3]{2x^2 - 2x + 1} + 4\sqrt[4]{3x^3 - 2x^4} = 2x^5 - 5x + 13$

$$2x^2 - 2x + 1 \ge 0, \Delta = -4 < 0, 3x^3 - 2x^4 \ge 0, x^3(3 - 2x) \ge 0$$

$$\Rightarrow 3 - 2x \ge 0 \Rightarrow x \le \frac{3}{2} \Rightarrow x \in \left[0, \frac{3}{2}\right]$$

$$\sqrt[3]{2x^2-2x+1}=\sqrt[3]{(2x^2-2x+1)\cdot 1\cdot 1}\underset{Mg\le Ma}{\le}\frac{2x^2-2x+1+1+1}{3}$$

$$\Rightarrow 6\sqrt[3]{2x^2-2x+1}\le 4x^2-4x+6$$

$$\sqrt[4]{3x^3-2x^4}=\sqrt[4]{(3x^3-2x^4)\cdot 1\cdot 1\cdot 1}\underset{Mg\le Ma}{\le}\frac{3x^3-2x^4+1+1+1}{4}$$

$$\Rightarrow 4\sqrt[4]{3x^3-2x^4}\le 3x^3-2x^4+3$$

$$\Rightarrow 2x^5-5x+13\le 4x^2-4x+6+3x^3-2x^4+3$$

$$\Rightarrow 2x^5+2x^4-3x^3-4x^2-x+4\le 0$$

$$(x^2-2x+1)(2x^3+6x^2+7x+4)\le 0$$

$$2x^3+6x^2+7x+4>0,(\forall)x\in\left[0,\frac{3}{2}\right]$$

$$\Rightarrow (x^2-2x+1)\le 0\Rightarrow (x-1)^2\le 0$$

$$(x-1)^2\ge 0\Rightarrow (x-1)^2=0\Rightarrow x-1=0; x=1=y$$

$$\Rightarrow S=\{(1,1)\}$$

3.5 Solve the following system of equations:

$$\begin{cases} \boldsymbol{x^3+2x+3=8y^3-6xy+4y} \\ \boldsymbol{\sqrt{x^2-2y+2}+\sqrt{x^2-4y+4}=x^2-3y+4} \end{cases}$$

Hoang Le Nhat Tung

Solution (Amit Dutta)

Domain $(x^2-2y+2)\ge 0,(x^2-4y+4)\ge 0$

$$GM\le AM$$

$$\sqrt{x^2-2y+2}=\sqrt{(x^2-2y+2)\cdot 1}\le\frac{(x^2-2y+2)+1}{2}$$

$$\sqrt{x^2-2y+2}\le\left(\frac{x^2-2y+3}{2}\right)$$

$$\sqrt{x^2-4y+4}=\sqrt{(x^2-4y+4)\cdot 1}\le\left(\frac{x^2-4y+5}{2}\right)$$

Adding these: $\sqrt{x^2-2y+2}+\sqrt{x^2-4y+4}\le(x^2-3y+4)$

But we have: $\sqrt{x^2-2y+2}+\sqrt{x^2-4y+4}=(x^2-3y+4)$

So, for equality, we must have: $\begin{cases} x^2-2y+2=1\Rightarrow x^2=2y-1 \\ x^2-4y+4=1\Rightarrow x^2=4y-3 \end{cases}$

Solve these two equations, we get: $\begin{Bmatrix} x=\pm 1 \\ y=1 \end{Bmatrix}$

But for the system of equation, we must check these solutions for the other equation also: i.e., $x^3+2x+3=8y^3-6xy+4y$

For $(x,y)=(1,1)$, $LHS=6$, $RHS=6$

Equality holds, so $(1,1)$ *is a solution for other possible solution:*

$(x,y)=(-1,1)$, $LHS=0$, $RHS=18$

Equality do not hold. So, $(-1,1)$ *is not the solution for this system of equation.*

Hence, $(1,1)$ *is the only solution.*

3.6 Find the positive real numbers (x,y) such that:

$$\begin{cases} \dfrac{x^2}{y}+\dfrac{y^2}{x}=2\sqrt[4]{\dfrac{x^4+y^4}{2}} \\ x^2y^2-y^3+1=\sqrt{2x^2-2y+1} \end{cases}$$

Hoang Le Nhat Tung

Solution (Soumava Chakraborty)

$$\sqrt{\frac{x^4+y^4}{2}}\le x^2-xy+y^2 \Leftrightarrow 2(x^2-xy+y^2)^2\ge x^4+y^4$$

$$\Leftrightarrow 2(x^4+y^4+2x^2y^2)-4xy(x^2+y^2)+2x^2y^2\ge x^4+y^4$$

$$\Leftrightarrow (x^2+y^2)^2-4xy(x^2+y^2)+4x^2y^2\ge 0 \Leftrightarrow$$

$$(x^2+y^2-2xy)^2\ge 0 \to true$$

$$\therefore 2\sqrt[4]{\frac{x^4+y^4}{2}}\le 2\sqrt{x^2-xy+y^2}\Rightarrow \frac{x^2}{y}+\frac{y^2}{x}\le 2\sqrt{x^2-xy+y^2}$$ *(by first equation)*

$$\Rightarrow \frac{(x+y)(x^2-xy+y^2)}{xy} \le 2\sqrt{x^2-xy+y^2} \Rightarrow (x+y)\sqrt{x^2-xy+y^2} \overset{(1)}{\le} 2xy$$

$$\text{But A-G} \Rightarrow (x+y)\sqrt{x^2-xy+y^2} \overset{(2)}{\ge} 2\sqrt{xy}\sqrt{xy} = 2xy$$

$$\therefore (1), (2) \Rightarrow (x+y)\sqrt{x^2-xy+y^2} = 2xy$$

and (2) suggests equality occurs when $x = y$

$$\therefore x = y \ (\because \text{equality occurs})$$

Putting $x = y$ in $x^2y^2 - y^3 + 1 = \sqrt{2x^2 - 2y + 1}$, we get:

$$x^4 - x^3 + 1 \overset{(3)}{=} \sqrt{2x^2 - 2x + 1}$$

Let $x^2 - x = p$. Then (3) $\Rightarrow x^2p + 1 = \sqrt{2p+1} \Rightarrow$

$$x^4p^2 + 1 + 2x^2p = 2p + 1 \Rightarrow p(x^4p + 2x^2 - 2) = 0$$

If $p = 0, x(x-1) = 0 \Rightarrow x = 0,1$ and both values satisfy (3), but $x, y \ne 0$

If $x^4p + 2x^2 - 2 = 0$, then: $x^4(x^2 - x) + 2(1+x)(x-1) = 0$

$$\Rightarrow x^5(x-1) + 2(x+1)(x-1) = 0 \Rightarrow (x-1)(x^5 + 2x + 2) = 0$$

$$\Rightarrow x = 1 \ (\because x^5 + 2x + 2 > 2 \ne 0 \text{ as } x > 0)$$

$\therefore$ combining all cases, all possible pairs of (x, y) satisfying given system is:

$$\begin{pmatrix} x = 1 \\ y = 1 \end{pmatrix} \text{ (answer)}$$

3.7 Solve for real numbers:

$$\begin{cases} \dfrac{1}{\sqrt{a}} + \dfrac{1}{\sqrt{b}} + \dfrac{1}{\sqrt{c}} = 3 \\ \dfrac{a^2}{b} + \dfrac{b^2}{c} + \dfrac{c^2}{a} = 3\sqrt[3]{\dfrac{a^3 + b^3 + c^3}{3}} \end{cases}$$

Hoang Le Nhat Tung

Solution (Rahim Shahbazov)

Lemma: If $a, b, c > 0$ then

$$\frac{a^2}{b}+\frac{b^2}{c}+\frac{c^2}{a}\geq 3\sqrt[4]{\frac{a^4+b^4+c^4}{3}}\geq 3\sqrt[3]{\frac{a^3+b^3+c^3}{3}}$$

Proof:

$$\sum\frac{a^2}{b}=\sum\frac{a^4}{a^2b}\overset{Bergstrom}{\geq}\frac{(a^2+b^2+c^2)^2}{a\cdot ab+b\cdot bc+c\cdot ca}$$

$$\overset{BCS}{\geq}\frac{(a^2+b^2+c^2)^2}{\sqrt{(a^2+b^2+c^2)((a^2b^2+b^2c^2+c^2a^2)}}\geq 3\sqrt[4]{\frac{a^4+b^4+c^4}{3}}$$

Then: $(a^2+b^2+c^2)^6\geq 27(a^2b^2+b^2c^2+c^2a^2)^2(a^4+b^4+c^4)$

Let: $a^2=x;\ b^2=y;\ c^2=z$ *then*

$(x+y+z)^6\geq 27(xy+yz+zx)^2(x^2+y^2+z^2)$ *true by*

$$(m+n+p)^3\overset{Am-Gm}{\geq}27mnp$$

We take: $m=x^2+y^2+z^2$ *and* $n=xy+yz+zx$ *then:*

$$\frac{a^2}{b}+\frac{b^2}{c}+\frac{c^2}{a}=3\sqrt[3]{\frac{a^3+b^3+c^3}{3}}\Rightarrow a=b=c=1.$$

$$S=\{(a,b,c)=(1,1,1)\}$$

3.8 Solve for real numbers:

$$\begin{cases}\mathbf{1\leq x,y,z\leq 3}\\ \mathbf{(x+y+z)\left(\frac{1}{x}+\frac{1}{y}+\frac{1}{z}\right)=\frac{35}{3}}\\ \mathbf{3^y+\log_2 z=3}\end{cases}$$

Daniel Sitaru

Solution (Ruangkhaw Chaoka)

$$(x+y+z)\left(\frac{1}{x}+\frac{1}{y}+\frac{1}{z}\right)=\frac{35}{3}\quad (1)$$

$$3^y+\log_2 z=3\quad (2)$$

We'll show that $(x+y+z)\left(\frac{1}{x}+\frac{1}{y}+\frac{1}{z}\right)\leq\frac{35}{3};1\leq x,y,z\leq 3$

$$Or\frac{x}{y}+\frac{y}{z}+\frac{z}{x}+\frac{y}{x}+\frac{z}{y}+\frac{x}{z}\overset{??}{\leq}\frac{26}{3}$$

$$WLOG\ x\geq y\geq z\Rightarrow 0\leq(x-y)(y-z)\Leftrightarrow zx+y^2\leq xy+yz$$

$$\begin{cases}\frac{z}{y}+\frac{y}{x}\leq 1+\frac{z}{x}\\ \frac{x}{y}+\frac{y}{z}\leq\frac{x}{z}+1\end{cases}\overset{+}{\Rightarrow}\frac{x}{y}+\frac{y}{z}+\frac{z}{y}+\frac{y}{x}\leq\frac{x}{z}+\frac{z}{x}+2$$

$$\frac{x}{y}+\frac{y}{z}+\frac{z}{x}+\frac{y}{x}+\frac{z}{y}+\frac{x}{z}\leq 2\left(\frac{x}{z}+\frac{z}{x}\right)+2$$

It remains to show that $2\left(\frac{x}{z}+\frac{z}{x}\right)+2\leq\frac{26}{3}$ *(3)*

Let $t=\frac{x}{z};1\leq t\leq 3\Leftrightarrow$ *(3);* $t+\frac{1}{t}\leq\frac{10}{3}\Leftrightarrow(3t-1)(t-3)\leq 0$

Which is true $\therefore$ *(1) is the hold point of inequality at* $t=\frac{x}{z}=3$

$$\begin{cases}x=3,z=1\Rightarrow y=1\\ x=3,z=1\Rightarrow y=3\end{cases}\text{ and permutations}\rightarrow\ (2)$$

$$\therefore(x,y,z)=(3,1,1)$$

3.9 Solve for positive real numbers:

$$\begin{cases}\frac{x^2}{y}+\frac{y^2}{x}=\sqrt[8]{128(x^8+y^8)}\\ 4x^3-3y=\sqrt{\frac{1+\sqrt{1-xy}}{2}}\end{cases}$$

Hoang Le Nhat Tung

Solution (Hoang Le Nhat Tung)

$$\begin{cases}\frac{x^2}{y}+\frac{y^2}{x}=\sqrt[8]{128(x^8+y^8)}\\ 4x^3-3y=\sqrt{\frac{1+\sqrt{1-xy}}{2}}\end{cases};(1)$$

By CBS inequality, we have:

$$\left(\sqrt{2(x^8+y^8)}+2x^2y^2\right)^2 \le 2(2(x^8+y^8)+4x^4y^4)=4(x^8+2x^4y^4+y^8)$$

$$=4(x^4+y^4)^2 \Rightarrow \sqrt{2(x^8+y^8)}+2x^2y^2 \le 2(x^4+y^4)$$

$$\Rightarrow \sqrt[4]{\frac{x^8+y^8}{2}} \le \sqrt{x^4-x^2y^2+y^4};(2)$$

Other,

$$\sqrt{x^4-x^2y^2+y^4}=$$

$$\sqrt{(2+\sqrt{3})(x^2-xy\sqrt{3}+y^2)(2-\sqrt{3})(x^2+xy\sqrt{3}+y^2)} \le$$

$$\le \frac{(2+\sqrt{3})(x^2-xy\sqrt{3}+y^2)+(2-\sqrt{3})(x^2+xy\sqrt{3}+y^2)}{2}$$

$$=2x^2-3xy+2y^2;(3).$$

From (2),(3) we have: $\sqrt[4]{\frac{x^8+y^8}{2}} \le 2x^2-3xy+2y^2;(4)$

By AM-GM we have:

$$\frac{x^2}{y}+\frac{y^2}{x}=\frac{x^3+y^3}{xy}=\frac{(x+y)[(2x^2-3xy+2y^2)+xy]}{2xy}$$

$$\ge \frac{2\sqrt{xy}\cdot 2\sqrt{xy(2x^2-3xy+2y^2)}}{2xy} \Rightarrow$$

$$\frac{x^2}{y}+\frac{y^2}{x} \ge 2\sqrt{xy(2x^2-3xy+2y^2)} \overset{(4)}{\Rightarrow} \frac{x^2}{y}+\frac{y^2}{x} \ge \sqrt[8]{128(x^8+y^8)};(5)$$

From (1),(5) we get:

$$\frac{x^2}{y}+\frac{y^2}{x}=\sqrt[8]{128(x^8+y^8)} \Leftrightarrow \begin{cases}\sqrt{2(x^8+y^8)}=2x^2y^2\\ x=y>0\end{cases} \Leftrightarrow x=y>0.$$

From (1) result: $4x^3-3x=\sqrt{\frac{1+\sqrt{1-x^2}}{2}}$.

Because: $0<x\le 1$, *put*: $x=\sin\alpha>0; \forall\alpha\in(0,\pi)$ *then*

$$4\sin^3\alpha-3\sin\alpha=\sqrt{\frac{1+\sqrt{1-\sin^2\alpha}}{2}} \Leftrightarrow \sin(-3\alpha)=\sqrt{\frac{1+|\cos\alpha|}{2}};(6)$$

Case 1. $cos\alpha \geq 0;\ \alpha \in (0,\pi) \Rightarrow \alpha \in [0,\frac{\pi}{2})$

$$(6) \Leftrightarrow sin(-3\alpha) = \sqrt{\frac{1+cos\alpha}{2}} = \sqrt{\frac{1+2cos^2\frac{\alpha}{2}-1}{2}} = \sqrt{cos^2\frac{\alpha}{2}} = cos\frac{\alpha}{2} \Leftrightarrow$$

$$sin(-3\alpha) = cos\frac{\alpha}{2} \Leftrightarrow sin(-3\alpha) = sin\left(\frac{\pi}{2}-\frac{\alpha}{2}\right)$$

$$\Rightarrow \begin{cases} -3\alpha = \frac{\pi}{2} + 2k\pi \\ -3\alpha = \pi - \left(\frac{\pi}{2}-\frac{\alpha}{2}\right) + 2k\pi \end{cases} \Leftrightarrow \begin{cases} \alpha = -\frac{\pi}{5} - \frac{4k\pi}{5} \\ \alpha = -\frac{\pi}{7} - \frac{4k\pi}{7} \end{cases}$$

If $\alpha = -\frac{\pi}{5} - \frac{4k\pi}{5}$, *because* $\alpha \in [0,\frac{\pi}{2})$ *we get:*

$$0 \leq -\frac{\pi}{5} - \frac{4k\pi}{5} < \frac{\pi}{2} \Leftrightarrow \frac{1}{4} \leq -k < \frac{7}{8} \Leftrightarrow -\frac{1}{4} \geq k > -\frac{7}{8};\ k \in \mathbb{Z} \text{ absurd!}$$

If $\alpha = -\frac{\pi}{7} - \frac{4k\pi}{7}$, *because* $\alpha \in [0,\frac{\pi}{2})$ *we get:*

$$0 \leq -\frac{\pi}{7} - \frac{4k\pi}{7} < \frac{\pi}{2} \Leftrightarrow \frac{1}{4} \leq -k < \frac{9}{8} \Leftrightarrow -\frac{1}{4} \geq k > -\frac{9}{8};\ k \in \mathbb{Z} \Rightarrow k = -1 \Rightarrow \alpha =$$

$$\frac{3\pi}{7} \Rightarrow x = y = sin\frac{3\pi}{7} = sin\frac{4\pi}{7}.$$

Case 2. $cos\alpha < 0; \alpha \in (0,\pi) \Rightarrow \alpha \in \left(\frac{\pi}{2},\pi\right)$

$$(6) \Leftrightarrow sin(-3\alpha) = \sqrt{\frac{1-cos\alpha}{2}} = \sqrt{\frac{1-\left(1-2sin^2\frac{\alpha}{2}\right)}{2}} = \sqrt{sin^2\frac{\alpha}{2}} = sin\frac{\alpha}{2}$$

$$\Leftrightarrow sin(-3\alpha) = sin\frac{\alpha}{2} \Leftrightarrow \begin{cases} -3\alpha = \frac{\alpha}{2} + 2k\pi \\ -3\alpha = \pi - \frac{\alpha}{2} + 2k\pi \end{cases} \Leftrightarrow \begin{cases} \alpha = -\frac{4k\pi}{7} \\ \alpha = -\frac{2k\pi}{5} - \frac{4k\pi}{5} \end{cases}$$

If $\alpha = -\frac{4k\pi}{7}$, *because* $\alpha \in \left(\frac{\pi}{2},\pi\right) \Rightarrow \frac{\pi}{2} < -\frac{4k\pi}{7} < \pi \Leftrightarrow \frac{7}{8} \leq -k < \frac{7}{4} \Leftrightarrow$

$-\frac{7}{8} > k > -\frac{7}{4};\ k \in \mathbb{Z}$ *then* $k = -1 \Rightarrow \alpha = \frac{4\pi}{7} \Rightarrow x = y = sin\frac{4\pi}{7}.$

If $\alpha = -\frac{2k\pi}{5} - \frac{4k\pi}{5}$, *because* $\alpha \in \left(\frac{\pi}{2},\pi\right) \Rightarrow \frac{\pi}{2} < -\frac{2\pi}{5} - \frac{4k\pi}{5} < \pi \Leftrightarrow$

$\frac{9}{8} < -k < \frac{7}{4} \Leftrightarrow -\frac{9}{8} > k > -\frac{7}{4};\ k \in \mathbb{Z}$ *absurd!. Therefore*:

$$(x,y)=\left(sin\frac{4\pi}{7};sin\frac{4\pi}{7}\right)$$

3.10 Solve for real numbers:

$$\begin{cases}\dfrac{x^2}{2y}+\dfrac{y^2}{2x}=\sqrt[4]{\dfrac{x^4+y^4}{2}}\\ x^2y^2-y^3+1=\sqrt{2x^2-2y+1}\end{cases}$$

Hoang Le Nhat Tung

Solution (Miguel Velasquez Culque)

Condition $x>0,y>0$ *and* $x\neq 0,y\neq 0$. *By AM-GM we have:*

$$\frac{x^2}{2y}+\frac{y^2}{2x}\geq 2\sqrt{\frac{x^2}{2y}\cdot\frac{y^2}{2x}}=\sqrt{xy}\Leftrightarrow\sqrt[4]{\frac{x^4+y^4}{2}}\geq\sqrt{xy}\Leftrightarrow\frac{x^4+y^4}{2}\geq x^2y^2\Leftrightarrow$$

$x^4+y^4\geq 2x^2y^2\Leftrightarrow(x^2-y^2)^2\geq 0$. *Equality holds if* $x=y$.

Replace in the second equation:

$$x^2y^2-y^3+1=\sqrt{2x^2-2y+1}\Rightarrow x^4-x^3+1=\sqrt{2x^2-2x+1}\Leftrightarrow$$

$$(x^4-x^3+1)^2=2x^2-2x+1\Leftrightarrow x^8-2x^7+x^5+2x^3-2x^2+2x=0\Leftrightarrow$$

$$x(x^7-2x^6+x^5+2x^3-2x^2-2x+2)=0\Leftrightarrow$$

$$x(x-1)^2(x^5+2x+2)=0\Leftrightarrow$$

$x=1$, *because* $x\neq 0$ *and* $x^5+2x+2>2>0$

3.11 Solve for real numbers:

$$\begin{cases}xy(4xy-1)^2+16xy=16z^2\\ yz(4yz-1)^2+16yz=16x^2\\ zx(4zx-1)^2+16zx=16y^2\end{cases}$$

Daniel Sitaru

Solution (Bedri Hajrizi)

It's clear that trivial solution is $(0,0,0)$.*Suppose that* $(x,y,z)\neq(0,0,0)$

Produce all we get: $\prod((4xy-1)^2+16)=16^3$

$$16^3 = \prod((4xy-1)^2+16) \geq 16^3 \Rightarrow \begin{cases} xy = \dfrac{1}{4} \\ yz = \dfrac{1}{4} \\ zx = \dfrac{1}{4} \end{cases}$$

Produce all $x^2y^2z^2 = \left(\frac{1}{4}\right)^3$ and $xy = \frac{1}{4} \Rightarrow x^2y^2z^2 = \frac{1}{16}z^2 \Rightarrow$

$\frac{1}{16}z^2 = \frac{1}{64} \Rightarrow z = \pm\frac{1}{2}$. *Similarly*: $x = y = \pm\frac{1}{2}$.

Finally solutions are:

$$(0,0,0), \left(-\frac{1}{2}, -\frac{1}{2}, -\frac{1}{2}\right), \left(\frac{1}{2}, \frac{1}{2}, \frac{1}{2}\right)$$

3.12 Suppose:

$$\begin{cases} \tan^{-1}\left(\dfrac{x+y}{1-xy}\right) = \sqrt[3]{\tan^{-1}z} \\ \tan^{-1}\left(\dfrac{y+z}{1-yz}\right) = \sqrt[3]{\tan^{-1}x} \\ \tan^{-1}\left(\dfrac{z+x}{1-xz}\right) = \sqrt[3]{\tan^{-1}y} \end{cases}$$

Find: $\boldsymbol{\Omega = x + y + z; x, y, z \in \mathbb{R}}$

Daniel Sitaru

Solution (Adrian Popa)

For $x, y, z \in \mathbb{R} \Rightarrow \exists a, b, c \in \left(-\frac{\pi}{2}, \frac{\pi}{2}\right)$ *such that:*

$$x = \tan a, y = \tan b, z = \tan c$$

$$\tan^{-1}\left(\frac{\tan a + \tan b}{1 - \tan a \cdot \tan b}\right) = \sqrt[3]{\tan^{-1}(\tan c)} \Rightarrow \begin{cases} a + b = \sqrt[3]{c}; (1) \\ b + c = \sqrt[3]{a}; (2) \\ c + a = \sqrt[3]{b}; (3) \end{cases}$$

$$(1) - (2) \Rightarrow a - c = \sqrt[3]{c} - \sqrt[3]{a}$$

$$\text{If } a > c \Rightarrow \begin{cases} a - c > 0 \\ \sqrt[3]{c} - \sqrt[3]{a} < 0 \end{cases} \Rightarrow a - c \neq \sqrt[3]{c} - \sqrt[3]{a}$$

$$\text{If } a < c \Rightarrow \begin{cases} a - c < 0 \\ \sqrt[3]{c} - \sqrt[3]{a} > 0 \end{cases} \Rightarrow a - c \neq \sqrt[3]{c} - \sqrt[3]{a} \Rightarrow a = c.$$

Similarly we get: $b = c; a = b \Rightarrow a = b = c \Rightarrow 2a = \sqrt[3]{a} \Rightarrow$

$$8a^3 = a \Rightarrow a(8a^2 - 1) = 0 \Rightarrow a \in \left\{-\frac{1}{2\sqrt{2}}, 0, \frac{1}{2\sqrt{2}}\right\}$$

$$(i) a = b = c = 0 \Rightarrow x = y = z = 0 \Rightarrow x + y + z = 0$$

$$(ii) a = b = c = \frac{1}{2\sqrt{2}} \Rightarrow x = y = z = tan\left(\frac{1}{2\sqrt{2}}\right) \Rightarrow x + y + z = 3tan\left(\frac{1}{2\sqrt{2}}\right)$$

$$(iii) a = b = c = -\frac{1}{2\sqrt{2}} \Rightarrow x = y = z = -tan\left(\frac{1}{2\sqrt{2}}\right) \Rightarrow x + y + z = -3tan\left(\frac{1}{2\sqrt{2}}\right)$$

3.13 Solve for complex numbers:

$$\begin{cases} \frac{1}{x} + \frac{1}{y} + \frac{1}{z} = 1 \\ \frac{x}{y} + \frac{y}{z} + \frac{z}{x} + \frac{x}{z} + \frac{y}{x} + \frac{z}{y} + 2 = 0 \end{cases}$$

Daniel Sitaru

Solution (Florentin Vişescu)

$$\begin{cases} \frac{1}{x} + \frac{1}{y} + \frac{1}{z} = 1 \\ \frac{x}{y} + \frac{y}{z} + \frac{z}{x} + \frac{x}{z} + \frac{y}{x} + \frac{z}{y} + 2 = 0 \end{cases}$$

$$\Rightarrow \begin{cases} \frac{1}{x} + \frac{1}{y} + \frac{1}{z} = 1 \\ x\left(\frac{1}{y} + \frac{1}{z}\right) + y\left(\frac{1}{z} + \frac{1}{x}\right) + z\left(\frac{1}{x} + \frac{1}{y}\right) + 2 = 0 \end{cases} \Rightarrow$$

$$\begin{cases} xy + yz + zx = xyz \\ x\left(1 - \frac{1}{x}\right) + y\left(1 - \frac{1}{y}\right) + z\left(1 - \frac{1}{z}\right) + 2 = 0 \end{cases}$$

$$\Rightarrow \begin{cases} xy + yz + zx = xyz = c; c \in \mathbb{C}^* \\ x - 1 + y - 1 + z - 1 + 2 = 0 \end{cases}$$

$$\Rightarrow \begin{cases} xy + yz + zx = xyz = c; c \in \mathbb{C}^* \\ x + y + z = 1 \end{cases}$$

But: x, y, z *—are solutions of the equation:*

$$t^3 - t^2 + ct - c = 0 \Leftrightarrow (t-1)(t^2+c) = 0$$

$$\Leftrightarrow t_1 = 1; t_2 = i\sqrt{|c|} = k \in \mathbb{C}^*; t_3 = -i\sqrt{|c|} = -k$$

So, $(x, y, z) \in \{(1; k; -k)_{cyc} \mid k \in \mathbb{C}^*\}$

3.14 Solve for real numbers:

$$\begin{cases} \boldsymbol{x^4 + 2y^3 - 6z^2 + 1 = 0, x, y, z > 0} \\ \boldsymbol{\frac{1}{42x + 43(y+z)} + \frac{1}{42y + 43(z+x)} + \frac{1}{42z + 43(x+y)} = \frac{1}{128}\left(\frac{1}{x} + \frac{1}{y} + \frac{1}{z}\right)} \end{cases}$$

Daniel Sitaru

Solution (Remus Florin Stanca)

We know that:

$$\frac{n}{\frac{1}{x_1} + \cdots + \frac{1}{x_n}} \le \frac{x_1 + \cdots + x_n}{n} \Rightarrow \frac{128}{42z + 43(x+y)} =$$

$$= \frac{128}{z + z + \cdots + z + x + x + \cdots + x + y + y + \cdots + y} \le \frac{\frac{42}{z} + \frac{43}{x} + \frac{43}{y}}{128}$$

$$(z - 42\ times, x - 43\ times, y - 53 - times) \Rightarrow$$

$$\frac{1}{42z + 43(x+y)} \le \frac{\frac{42}{z} + 43\left(\frac{1}{x} + \frac{1}{y}\right)}{128^2} \Rightarrow$$

$$\sum_{cyc} \frac{1}{42z + 43(x+y)} \le \frac{128\left(\frac{1}{x} + \frac{1}{y} + \frac{1}{z}\right)}{128^2} = \frac{1}{128}\left(\frac{1}{x} + \frac{1}{y} + \frac{1}{z}\right)$$

But $\sum_{cyc} \frac{1}{42z+43(x+y)} = \frac{1}{128}\left(\frac{1}{x} + \frac{1}{y} + \frac{1}{z}\right) \Rightarrow x = y = z \Rightarrow$

$$x^4 + 2x^3 - 6x^2 + 1 = 0 \Rightarrow$$

$$x^4 - x^3 + 3x^3 - 3x^2 + 3x - x + 1 = 0 \Rightarrow$$

$$x^3(x-1) + 3x(x^2-1) - 3x(x-1) - (x-1) = 0 \Rightarrow$$

$$(x-1)(x^2 + 3x^2 - 3x - 1) = 0 \Rightarrow (x-1)^2(x^2 + 4x + 1) = 0 \Rightarrow x = 1$$

$$\Rightarrow (x; y; z) \in \{(1; 1; 1)\}$$

COMPLEX NUMBERS

4.1 Let be $A(z_1); B(z_1); C(z_3); z_1, z_2, z_3 \in \mathbb{C} \setminus \{0\};$

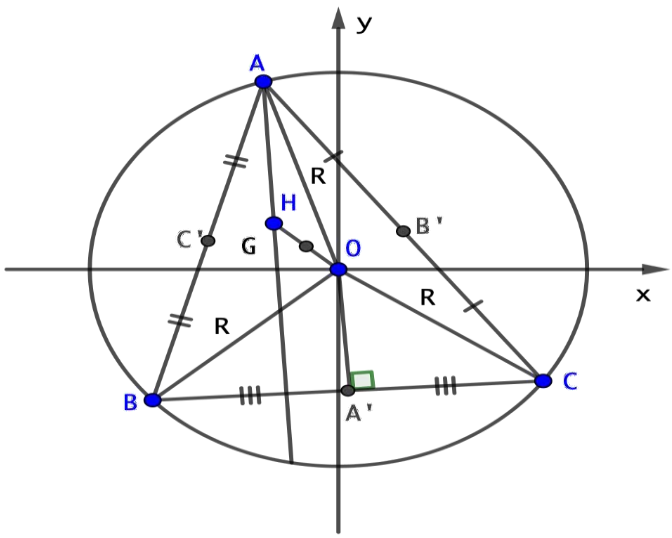
$|z_1| = |z_2| = |z_3|; AB = c; BC = a; CA = b.$

If $(b + c)z_B z_C + (c + a)z_C z_A + (a + b)z_A z_B = 0$ **then**

$AB = BC = CA.$

Marian Ursărescu

Solution (Khaled Abd Imouti)



$$Let\ be\ A(z_1), B(z_2), C(z_3), z_1, z_2, z_3 \in \mathbb{C} \setminus \{0\}$$

$$|z_1| = |z_2| = |z_3|, AB = c, BC = a, CA = b$$

$$If\ (b + c) \cdot z_B \cdot z_C + (c + a) \cdot z_C \cdot z_A + (a + b) \cdot z_A \cdot z_B = 0$$

$$Then\ AB = BC = CA.$$

$$|z_A| = |z_B| = |z_C| = R\ (R\ radius\ of\ circle)$$

$$z_A \cdot \overline{z_A} = R^2, z_B \cdot \overline{z_B} = R^2, z_C \cdot \overline{z_C} = R^2$$

$$z_A = \frac{R^2}{\overline{z_A}}, z_B = \frac{R^2}{\overline{z_B}}, z_C = \frac{R^2}{\overline{z_C}}$$

$$(b+c)\cdot\left(\frac{R^2}{\overline{z_B}}\cdot\frac{R^2}{\overline{z_C}}\right)+(c+a)\left(\frac{R^2}{\overline{z_C}}\cdot\frac{R^2}{\overline{z_A}}\right)+(a+b)\left(\frac{R^2}{\overline{z_A}}\cdot\frac{R^2}{\overline{z_B}}\right)=0$$

$$(b+c)\cdot\left(\frac{1}{\overline{z_B}\cdot\overline{z_C}}\right)+(c+d)\left(\frac{1}{\overline{z_C}\cdot\overline{z_A}}\right)+(a+b)\left(\frac{1}{\overline{z_A}\cdot\overline{z_C}}\right)=0$$

$$\frac{(b+c)}{z_B\cdot z_C}+\frac{(c+a)}{z_C\cdot z_A}+\frac{a+b}{z_A\cdot z_B}=0\ \times(z_A\cdot z_B\cdot z_C\neq 0)$$

$$(b+c)z_A+(c+a)z_B+(a+b)\cdot z_C=0$$

$$(b+c+a-a)z_A+(c+a+b-b)z_B+(a+b+c-c)z_C=0$$

$$2p(z_A+z_B+z_C)=a\cdot z_A+b\cdot z_B+c\cdot z_C$$

$$6p\cdot z_G=a\cdot z_A+b\cdot z_B+c\cdot z_C,\qquad z_G=\frac{a\cdot z_A+b\cdot z_B+c\cdot z_C}{6p}$$

Suppose H is orthocenter: $\overrightarrow{OH}=\overrightarrow{OA}+\overrightarrow{OB}+\overrightarrow{OC}$ *(*)*

$$\overrightarrow{OH}=\overrightarrow{OA}+\left(\overrightarrow{OB}+\overrightarrow{OC}\right)\Rightarrow\overrightarrow{ON}-\overrightarrow{OA}=2\overrightarrow{OA'}$$

So: $\overrightarrow{AH}=2\cdot\overrightarrow{OA'},(AH)\parallel(OA')$ *but* $OA'\perp BC\Rightarrow AH\perp BC$

in a similar way, $BH\perp AC$ *and* $CH\perp AB$*so,* H *is orthocenter in triangle* ΔABC*, from (*):* $\overrightarrow{OH}=3\overrightarrow{OG}$*, so:* O,H,G *is collinear.*

not: $z_H=3\cdot z_G$

$$z_G-z_H=\frac{a\cdot z_A+b\cdot z_B+c\cdot z_C}{6p}-\frac{z_A+z_B+z_C}{1}$$

$$z_G-z_H=\frac{(a-6p)z_A+(6-6p)z_B+(c-6p)z_C}{6p}$$

$$-z_G+z_H=\frac{(6p-a)z_A+(6p-b)z_B+(6p-c)z_C}{6p}$$

$$6p(z_H-z_G)=(6p-a)z_A+(6p-b)z_B+(6p-c)z_C$$

$$\frac{6p(z_H-z_G)}{16p}=\frac{(6p-a)z_A+(6p-b)z_B+(6p-c)z_C}{16p}$$

$$\frac{3}{9}(z_H-z_G)=\frac{1}{16p}[(6p-a)z_A+(6p-b)z_B+(6p-c)z_C]$$

$$\text{but: } (6p-a)z_A+(6p-b)z_B+(6p-c)z_C=6p(z_A+z_B+z_C)-(az_A+bz_B+cz_C)$$

$$(6p-a)z_A+(6p-b)z_B+(6p-c)z_C=18p\cdot\left(\frac{z_A+z_B+z_C}{3}\right)-(a\cdot z_A+b\cdot z_B+c\cdot z_C)$$

$$(6p-a)\cdot z_A+(6p-b)z_B+(6p-c)z_C=18p\cdot z_G-(a\cdot z_A+b\cdot z_B+c\cdot z_C)$$

$$\frac{(6p-a)z_A+(6p-b)z_B+(6p-c)z_C}{16p}=\frac{18p\cdot z_G}{16p}-\frac{a\cdot z_A+b\cdot z_B+c\cdot z_C}{16p}$$

$$=\frac{9}{4}z_G-\frac{1}{8}\left(\frac{a\cdot z_A+b\cdot z_B+c\cdot z_C}{2p}\right)=\frac{9}{4}z_G-\frac{1}{8}\cdot z_H$$

$$\text{So: } \frac{3}{8}(z_H-z_G)=\frac{9}{4}z_G-\frac{1}{8}z_H,\ \ \frac{3}{8}z_H+\frac{1}{8}z_H=\frac{9}{4}z_G+\frac{3}{8}z_G$$

$$\frac{1}{2}z_H=\frac{21}{8}z_G\Rightarrow z_H=\frac{21}{4}z_G \text{ and: } z_H=3\cdot z_G$$

$$\frac{21}{4}z_G-3z_G=0,\ \ \frac{9}{4}z_G=0\Rightarrow z_G=0$$

$G\equiv O$, So ΔABC is equilateral triangle.

4.2 If $x,y,z\in C, x+y+z=3+4i$ then:

$$|x-z|+|y-x|+|z-y|+5\ge 3\sqrt[3]{|xyz|}$$

Daniel Sitaru

Solution (Tran Hong)

Note: $|x|$: module of $x\in\mathbb{C}$

$$x+y+z=3+4i\to|x+y+z|=|3+4i|=\sqrt{3^2+4^2}=5$$

$$|x-z|+|y-x|+|z-y|\ge 0\to LHS=|x-z|+|y-x|+|z-y|+5$$

$$=|x-z|+|y-x|+|z-y|+|x+y+z|\overset{(1)}{\ge}|x|+|y|+|z|$$

We prove that (1) is true with for all $x,y,z\in\mathbb{C}$

In fact, by Hlawka's Inequality for complex a,b,c:

$$|a+b|+|b+c|+|c+a|\le|a|+|b|+|c|+|a+b+c| \ (*)$$

In (*) we choose: $x = a + b; y = b + c; z = c + a$

$$\to x + y + z = 2(a + b + c) \to a + b + c = \frac{x + y + z}{2} \to a = \frac{x + z - y}{2};$$

$$b = \frac{x + y - z}{2}; c = \frac{y + z - x}{2}$$

$$(*) \leftrightarrow |x| + |y| + |z| \le \left|\frac{x+z-y}{2}\right| + \left|\frac{x+y-z}{2}\right| + \left|\frac{y+z-x}{2}\right| + \left|\frac{x+y+z}{2}\right|$$

$$\le \left|\frac{x - y}{2}\right| + \left|\frac{z}{2}\right| + \left|\frac{y - z}{2}\right| + \left|\frac{x}{2}\right| + \left|\frac{x - z}{2}\right| + \left|\frac{y}{2}\right| + \left|\frac{x + y + z}{2}\right| =$$

$$\frac{|x| + |y| + |z|}{2} + \left|\frac{x - y}{2}\right| + \left|\frac{y - z}{2}\right| + \left|\frac{x - z}{2}\right| + \left|\frac{x + y + z}{2}\right|$$

$$\leftrightarrow \frac{|x| + |y| + |z|}{2} \le \frac{|x - z| + |y - x| + |z - y|}{2} + \frac{|x + y + z|}{2}$$

$$\leftrightarrow |x - z| + |y - x| + |z - y| + |x + y + z| \ge |x| + |y| + |z| \to \textit{(1) is true.}$$

Because: $|x|; |y|; |z| \ge 0$. *Using AM-GM we have:*

$$|x| + |y| + |z| \ge 3\sqrt[3]{|x| \cdot |y| \cdot |z|} = 3\sqrt[3]{|xyz|} = RHS. \textit{ Proved.}$$

4.3 Find all positive real numbers (x, y, z) such that:

$$\begin{cases} x^2 + y^2 + z^2 = 3 \\ x^3y + y^3z + z^3x = \dfrac{2x}{y^2 + z^2} + \dfrac{2y}{z^2 + x^2} + \dfrac{2z}{x^2 + y^2} \end{cases}$$

Hoang Le Nhat Tung

Solution (Ruangkhaw Chaoka)

$$x^2 + y^2 + z^2 = 3 \quad (a)$$

$$x^3y + y^3z + z^3x = \frac{2x}{y^2+z^2} + \frac{2y}{z^2+x^2} + \frac{2z}{x^2+y^2} \quad (b)$$

Lemma. $\frac{2x}{y^2+z^2} + \frac{2y}{z^2+x^2} + \frac{2z}{x^2+y^2} \ge 3$ *(1) (hold at* $x = y = z = 1$*)*

Proof. $\frac{2x}{y^2+z^2} = \frac{2x}{3-x^2} \overset{??}{\ge} x^2 \Leftrightarrow x^4 - 3x^2 + 2x = x(x - 1)^2(x + 2) \ge 0$ *true*

(hold at $x = 1$*)*

Similarly: $\frac{2y}{z^2+x^2} \ge y^2$ *(holds at* $y = 1$*) and* $\frac{2z}{x^2+y^2} \ge z^2$ *(holds at* $z = 1$*)*

$\therefore \frac{2x}{y^2+z^2}+\frac{2y}{z^2+x^2}+\frac{2z}{x^2+y^2} \geq x^2+y^2+z^2=3$ *(holds at* $x=y=z=1$*)*

(Vasile Cîrtoaje, 1992) $(x^2+y^2+z^2)^2 \geq 3(x^3y+y^3z+z^3x)$

(holds at $x^2-xy+yz=y^2-yz+zx=z^2-zx+xy$*)*

$$\frac{1}{2}\left\{\begin{matrix}[(x^2-xy+yz)-(y^2-yz+zx)]^2+[(y^2-yz+zx)-(z^2-zx+xy)]^2+\\ +[(z^2-zx+xy)-(x^2-xy+yz]^2\end{matrix}\right\} \geq 0 \quad (2)$$

(1), (2): $\frac{2x}{y^2+z^2}+\frac{2y}{z^2+x^2}+\frac{2z}{x^2+y^2} \geq 3=\frac{(x^2+y^2+z^2)^2}{3} \geq (x^3y+y^3z+z^3x)$

(holds at $x=y=z=1$*)*

$\therefore$ *(b) is the hold point of inequality at* $x=y=z=1$

4.4 $\boldsymbol{\Omega_1=|z_1+z_2+z_3|, z_1,z_2,z_3 \in \mathbb{C}}$

$$\boldsymbol{\Omega_2=|z_1+z_2-z_3+4i|+|z_1-z_2+z_3+2i|+|-z_1+z_2+z_3-6i|}$$

Prove that: $\boldsymbol{\Omega_1 \leq \Omega_2}$

Daniel Sitaru

Solution (Ravi Prakash)

$$\Omega_1=|z_1+z_2+z_3|$$

$$\Omega_2=|z_1+z_2-z_3+4i|+|z_1-z_2+z_3+2i|+|-z_1+z_2+z_3-6i|$$

$$\geq |z_1+z_2-z_3+4i+z_1-z_2+z_3+2i-z_1+z_2+z_3-6i|$$

$$=|z_1+z_2+z_3|=\Omega_1, \qquad \therefore \Omega_2 \geq \Omega_1$$

4.5 Let be $\boldsymbol{z_A, z_B, z_C \in \mathbb{C}^*}$**, different in pairs such that**

$\boldsymbol{|z_A|=|z_B|=|z_C|=1}$**. If**

$\boldsymbol{|z_A-z_B-z_C|+|z_B-z_C-z_A|+|z_C-z_A-z_B|=6}$**, then** $\boldsymbol{\Delta ABC}$ **is an equilateral triangle.**

Marian Ursărescu

Solution

Let $A(z_1), B(z_2), C(z_3), \Delta ABC \subset C(0,1)$ and $\Omega -$ the middle of OH $-$(point by Euler)

$$z_\Omega = \frac{z_O + z_H}{2} = \frac{z_1 + z_2 + z_3}{2} \Rightarrow A\Omega = |z_A - z_\Omega| = \left|z_1 - \frac{z_1 + z_2 + z_3}{2}\right| = \frac{|z_1 - z_2 - z_3|}{2}$$

$$|z_1 - z_2 - z_3| + |z_2 - z_1 - z_3| + |z_3 - z_1 - z_2| = 6 \Leftrightarrow$$
$$A\Omega + B\Omega + C\Omega = 3;\ \ (1)$$

Let A' –midle of $BC \Rightarrow \Omega A'^2 = \frac{2(B\Omega^2 + C\Omega^2) - a^2}{4} \Rightarrow R^2 = 2(B\Omega^2 + C\Omega^2) - a^2$ and analogs.

$$A\Omega^2 + B\Omega^2 + C\Omega^2 = \frac{3R^2 + a^2 + b^2 + c^2}{4} \le \frac{3R^2 + 9R^2}{4} \le 3R^2$$

But: $(A\Omega + B\Omega + C\Omega)^2 \le 3(A\Omega^2 + B\Omega^2 + C\Omega^2) \le 9R^2;\ \ (2)$

From $(1), (2)$ equality when the ΔABC is equilateral.

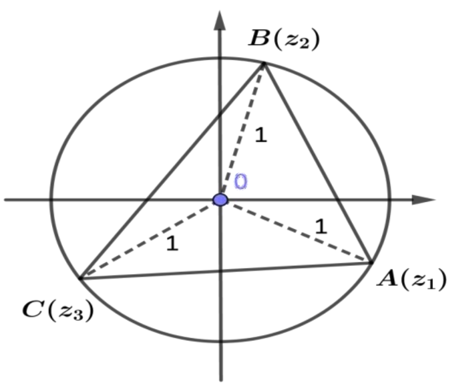
4.6 Let be $z_1, z_2, z_3 \in \mathbb{C} \setminus \{0\}$ different in pairs:

$$|z_1| = |z_2| = |z_3| = 1; A(z_1); B(z_2); C(z_3).$$

If $|z_1 - z_2 - z_3| + |z_2 - z_1 - z_3| + |z_3 - z_2 - z_1| = 6$ then $AB = BC = CA$.

Marian Ursărescu

Solution (Khaled Abd Imouti)



$$|z_1| = |z_2| = |z_3| = 1, A(z_1), B(z_2), C(z_3)$$
$$|z_1 - z_2 - z_3| + |z_2 - z_1 - z_3| + |z_3 - z_2 - z_1| = 6$$
$$|z_1 - (z_2 + z_3)| + |z_2 - (z_1 + z_3)| + |z_3 - (z_2 + z_1)| = 6$$

$$|z_1-(z_1+z_2+z_3-z_1)|+|z_2-(z_1+z_2+z_3-z_2)|$$
$$+|z_3-(z_1+z_2+z_3-z_3)|=6$$
$$|2z_1-3z_G|+|2z_2-3z_G|+|2z_3-3z_G|=6$$
$$|2z_1-3z_G|+|2z_2-3z_G|+|2z_3-3z_G|=2|z_1|+2|z_2|+2|z_3|$$
$$(|2z_1-3z_G|-2|z_1|)+(|2z_2-3z_G|-2|z_2|)+(|2z_3-3z_G|-2|z_3|)=0$$
(*)

Suppose $|2z_3-3z_G|\neq 2$ *and* $|2z_2-3z_G|\neq 2, |2z_3-3z_G|\neq 2$

$$(2z_3-3z_G)(2\overline{z_3}-3\overline{z_G})\neq 4, 4-6z_3\overline{z_G}-6z_G\overline{z_3}+9z_G\overline{z_G}\neq 4$$
$$2(z_3\cdot\overline{z_G}+z_G\cdot\overline{z_3})\neq 3\cdot z_G\cdot\overline{z_G}\quad (I)$$

Similarly, $2(z_1\overline{z_G}+z_G\overline{z_1})\neq 3z_G\cdot\overline{z_G}$ *(II)*

and $2(z_2\overline{z_G}+z_G\cdot\overline{z_2})\neq 3\cdot z_G\cdot\overline{z_G}$ *(III)*

By adding (I), (II), (III): $6\overline{z_G}\cdot z_G+6z_G\cdot\overline{z_G}+6z_G\overline{z_G}\neq 9z_G\cdot\overline{z_G}$

$$18z_G\cdot\overline{z_G}-9z_G\cdot\overline{z_G}\neq 0$$
$$9z_G\cdot\overline{z_G}\neq 0\Rightarrow 9|z_G|^2\neq 0\Rightarrow |z_G|\neq 0$$

$z_G\neq 0, G\not\equiv 0$ *and hence:* $|2z_1-3z_G|-2|z_1|\neq 0, |2z_3-3z_G|-2|z_2|\neq 0$ *and* $|2z_3-3z_G|-2|z_3|\neq 0$ *this is in contradiction with relation (*)*

So: $z_G=0\Rightarrow G\equiv 0$ *and the triangle ABC is equilateral.*

4.7 If $a,b\in\mathbb{C}$ then for any $z_1,z_2,z_3\in\mathbb{C}$ the following relationship holds:

$$|z_1+a+b|+|z_2-a|+|z_3-b|\le$$
$$\le|z_1+z_2+2b-z_3|+|z_1+z_3+a-z_2+a|+$$
$$+|-z_1+z_2+z_3-2a-2b|$$

Daniel Sitaru

Solution:

$|z_1+z_2-z_3+2b|+|z_1-z_2+z_3+2a|\ge|2z_1+2a+2b|$ (1)

$|z_1+z_2-z_3+2b|+|-z_1+z_2+z_3-2a-2b|\ge|2z_2-2a|$ (2)

$|z_1+z_3-z_2+2a|+|-z_1+z_2+z_3-2a-2b|\ge|2z_3-2b|$ (3)

By adding (1); (2); (3): $2RHS \geq 2LHS \Rightarrow LHS \leq RHS$

4.8 $\boldsymbol{z_A, z_B, z_C \in \mathbb{C}^*}$ **−differrent in pairs,** $\boldsymbol{|z_A| = |z_B| = |z_C| = 1}$

$\boldsymbol{a = BC, b = CA, c = AB}$**. Prove that:**

$$\left|\prod_{cyc} b(z_A - z_B) + c(z_A - z_C)\right| = (a+b+c)^3 \Rightarrow AB = BC = CA$$

Marian Ursărescu

Solution (Florentin Vişescu)

$$\left|\prod_{cyc} b(z_A - z_B) + c(z_A - z_C)\right| = (a+b+c)^3, |z_A| = |z_B| = |z_C| = 1 \Leftrightarrow$$

$$\prod_{cyc}\left|\frac{b(z_A - z_B) + c(z_A - z_C)}{a+b+c}\right| = 1 \Rightarrow \prod_{cyc}\left|\frac{(a+b+c)z_A}{a+b+c} - z_I\right| = 1 \Rightarrow$$

$$\prod_{cyc}|z_A - z_I| = 1 \Rightarrow |z_A - z_I| \cdot |z_B - z_I| \cdot |z_C - z_I| = 1 \Rightarrow AI \cdot BI \cdot CI = 1$$

$$\Rightarrow \frac{r^3}{sin\frac{A}{2}sin\frac{B}{2}sin\frac{C}{2}} = 1 \Rightarrow r^3 = sin\frac{A}{2}sin\frac{B}{2}sin\frac{C}{2} \Rightarrow$$

$$r^3 = \frac{(s-a)(s-b)(s-c)}{abc} \Rightarrow r^3 = \frac{S^2}{4RsS} \Rightarrow r^3 = \frac{S}{4Rs} \Rightarrow r^3 = \frac{rs}{4Rs}$$

$$\Rightarrow r^2 = \frac{1}{4R} \overset{R=1}{\Longrightarrow} r = \frac{R}{2} \Rightarrow AB = BC = CA$$

ABSTRACT ALGEBRA

5.1 Find all the polynomials $\boldsymbol{P} \in \mathbb{R}[\boldsymbol{x}]$ having the property

$$\boldsymbol{P}(\boldsymbol{x}) = \boldsymbol{P}\left(\boldsymbol{x} + \sqrt{\boldsymbol{x}^2 + \mathbf{1}}\right), \forall \boldsymbol{x} \in \mathbb{R}$$

Marian Ursărescu

Solution (Ravi Prakash)

Let $P(x)$ be a polynomial of degree m where $m \in \mathbb{N}$.

If $m = 1$, let $P(x) = ax + b, a \neq 0$, then

$$ax + b = a\left(x + \sqrt{x^2 + 1}\right) + b \ \ \forall x \in \mathbb{R}$$

$\Rightarrow a\sqrt{x^2 + 1} = 0, \forall x \in \mathbb{R} \Rightarrow a = 0$. *A contradiction. Assume $m \geq 2$.*

Choose a sequence $m_1 > m_2 > \cdots > m_m$ of positive integers such that

$$m_{k+1} > m_k + \sqrt{m_k^2 + 1} \text{ for } 1 \leq k \leq m - 1.$$

For $1 \leq r \leq m$

$$P(m_r) = P\left(m_r + \sqrt{m_r^2 + 1}\right) \ \ \text{(given)}$$

By the Rolle's theorem $\exists \alpha_r \in \left(m_r, m_r + \sqrt{m_r^2 + 1}\right)$ such that

$P'(\alpha_r) = 0 \quad (1 \leq r \leq m) \Rightarrow P'(x)$ *has at least m zeros. But $P'(x)$ is a polynomial of degree $(m - 1)$. A contradiction.*

$\therefore$ *there is no polynomial of degree ≥ 1, satisfying the given condition.*

Thus, $P(x)$ satisfies the given condition if and only if $P(x)$ is a constant

5.2 If $\boldsymbol{a}, \boldsymbol{b} \in \mathbb{R}; \boldsymbol{A} = \begin{pmatrix} \sin^2 a & \cos^2 a \sin^2 b & \cos^2 a \cos^2 b \\ \cos^2 b \sin^2 c & \sin^2 b & \cos^2 b \cos^2 c \\ \cos^2 c \sin^2 a & \cos^2 c \cos^2 a & \sin^2 c \end{pmatrix}$

$$\boldsymbol{A}^{\boldsymbol{n}} = \begin{pmatrix} x_1 & x_2 & x_3 \\ x_4 & x_5 & x_6 \\ x_7 & x_8 & x_9 \end{pmatrix}; \boldsymbol{n} \in \mathbb{N}; \boldsymbol{n} \geq \mathbf{2}; \boldsymbol{x}_{\boldsymbol{i}} \in \mathbb{R}; \boldsymbol{i} \in \overline{\mathbf{1}, \mathbf{9}}$$

then find $\Omega = \sum_{i=1}^{9} x_i$

Daniel Sitaru

Solution

$$A\cdot\begin{pmatrix}1\\1\\1\end{pmatrix}=\begin{pmatrix}\sin^2 a & \cos^2 a\sin^2 b & \cos^2 a\cos^2 b\\ \cos^2 b\sin^2 c & \sin^2 b & \cos^2 b\cos^2 c\\ \cos^2 c\sin^2 a & \cos^2 c\cos^2 a & \sin^2 c\end{pmatrix}\begin{pmatrix}1\\1\\1\end{pmatrix}=$$

$$=\begin{pmatrix}\sin^2 a+\cos^2 a\sin^2 b+\cos^2 a\cos^2 b\\ \cos^2 b\sin^2 c+\sin^2 b+\cos^2 b\cos^2 c\\ \cos^2 c\sin^2 a+\cos^2 c\cos^2 a+\sin^2 c\end{pmatrix}=$$

$$=\begin{pmatrix}\sin^2 a+\cos^2 a\,(\sin^2 b+\cos^2 b)\\ \sin^2 b+\cos^2 b\,(\sin^2 c+\cos^2 c)\\ \sin^2 c+\cos^2 c\,(\sin^2 a+\cos^2 a)\end{pmatrix}=\begin{pmatrix}\sin^2 a+\cos^2 a\\ \sin^2 b+\cos^2 b\\ \sin^2 c+\cos^2 c\end{pmatrix}=\begin{pmatrix}1\\1\\1\end{pmatrix}$$

$$A^2\cdot\begin{pmatrix}1\\1\\1\end{pmatrix}=A\cdot A\begin{pmatrix}1\\1\\1\end{pmatrix}=A\begin{pmatrix}1\\1\\1\end{pmatrix}=\begin{pmatrix}1\\1\\1\end{pmatrix}$$

$$P(n): A^n\begin{pmatrix}1\\1\\1\end{pmatrix}=\begin{pmatrix}1\\1\\1\end{pmatrix}\ \textit{(suppose true)}$$

$$P(n+1): A^{n+1}\begin{pmatrix}1\\1\\1\end{pmatrix}=\begin{pmatrix}1\\1\\1\end{pmatrix}\ \textit{(to prove)}$$

$$A^{n+1}\begin{pmatrix}1\\1\\1\end{pmatrix}=A\cdot A^n\begin{pmatrix}1\\1\\1\end{pmatrix}=A\begin{pmatrix}1\\1\\1\end{pmatrix}=\begin{pmatrix}1\\1\\1\end{pmatrix}$$

$$P(n)\to P(n+1)$$

$$A^n\cdot\begin{pmatrix}1\\1\\1\end{pmatrix}=\begin{pmatrix}1\\1\\1\end{pmatrix}\Rightarrow\begin{pmatrix}x_1 & x_2 & x_3\\ x_4 & x_5 & x_6\\ x_7 & x_8 & x_9\end{pmatrix}\cdot\begin{pmatrix}1\\1\\1\end{pmatrix}=\begin{pmatrix}1\\1\\1\end{pmatrix}$$

$$\Rightarrow\begin{pmatrix}x_1+x_2+x_3\\ x_4+x_5+x_6\\ x_7+x_8+x_9\end{pmatrix}=\begin{pmatrix}1\\1\\1\end{pmatrix}\Rightarrow\sum_{i=1}^{9}x_i=9\Rightarrow\Omega=9$$

5.3 Solve the system of equation: $\begin{cases} \dfrac{1}{\sqrt{a^3}}+\dfrac{1}{\sqrt{b^3}}+\dfrac{1}{\sqrt{c^3}}=3 \\ \dfrac{a^2}{b}+\dfrac{b^2}{c}+\dfrac{c^2}{a}=\dfrac{\left(a^3+b^3+c^3\right)^2}{3} \end{cases}$ **(1)**

Hoang Le Nhat Tung

Solution(Hoang Le Nhat Tung)

** Hence (1), by AM-GM inequality for three positive real numbers we have:*

$$3=\frac{1}{\sqrt{a^3}}+\frac{1}{\sqrt{b^3}}+\frac{1}{\sqrt{c^3}}\geq 3\cdot\sqrt[3]{\frac{1}{\sqrt{a^3}\cdot\sqrt{b^3}\cdot\sqrt{c^3}}}=\frac{3}{\sqrt[6]{(abc)^3}}=\frac{3}{\sqrt{abc}}\Leftrightarrow 3$$
$$\geq\frac{3}{\sqrt{abc}}\Leftrightarrow\sqrt{abc}\geq 1\Leftrightarrow abc\geq 1$$

Hence (2):

$$\Rightarrow\frac{a^2}{b}+\frac{b^2}{c}+\frac{c^2}{a}=\frac{a^3c+b^3a+c^3b}{abc}\leq\frac{a^3c+b^3a+c^3b}{1}=a^3c+b^3a+c^3b \quad (3)$$

- By AM-GM inequality, we have:

$$a^3c+b^3a+c^3b=a^3ac+b^2ba+c^2cb\leq\frac{a^3+(ac)^2}{2}+\frac{b^4+(ba)^2}{2}+\frac{c^4+(cb)^2}{2}$$
$$\Leftrightarrow a^3c+b^3a+c^3b\leq\frac{a^4+b^4+c^4+a^2b^2+b^2c^2+c^2a^2}{2} \quad (4)$$

- Hence (3), (4): $\Rightarrow\frac{a^2}{b}+\frac{b^2}{c}+\frac{c^2}{a}\leq\frac{a^4+b^4+c^4+a^2b^2+b^2c^2+c^2a^2}{2}$ *(5)*

- Other, by AM-GM inequality:

$$\frac{a^6+a^6+1}{2}+\frac{b^6+b^6+1}{2}+\frac{c^6+c^6+1}{2}$$
$$\geq\frac{3\sqrt[3]{a^6\cdot a^6\cdot 1}}{2}+\frac{3\sqrt[3]{b^6\cdot b^6\cdot 1}}{2}+\frac{3\sqrt[3]{c^6\cdot c^6\cdot 1}}{2}$$
$$=\frac{3(a^4+b^4+c^4)}{2}$$
$$\Leftrightarrow a^6+b^6+c^6+\frac{3}{2}\geq\frac{3(a^4+b^4+c^4)}{2}\Leftrightarrow$$
$$2(a^6+b^6+c^6)+3\geq 3(a^4+b^4+c^4) \quad (6)$$

$$(a^3b^3+a^3b^3+1)+(b^3c^3+b^3c^3+1)+(c^3a^3+c^3a^3+1)\geq$$

$$\geq 3\sqrt[3]{(a^3b^3)(a^3b^3)\cdot 1}+3\sqrt[3]{(b^3c^3)(b^3c^3)\cdot 1}+3\sqrt[3]{(c^3a^3)(c^3a^3)\cdot 1}$$

$$=3(a^2b^2+b^2c^2+c^2a^2)$$

$$\Leftrightarrow 2(a^3b^3+b^3c^3+c^3a^3)+3\geq 3(a^2b^2+b^2c^2+c^2a^2)$$

$$\Leftrightarrow 4(a^3b^3+b^3c^3+c^3a^3)+6\geq 6(a^2b^2+b^2c^2+c^2a^2) \quad (7)$$

- Let (6), (7): $\Rightarrow 2(a^6+b^6+c^6)+3+4(a^3b^3+b^3c^3+c^3a^3)+6\geq$

$$\geq 3(a^4+b^4+c^4)+6(a^2b^2+b^2c^2+c^2a^2)$$

$$\Leftrightarrow 2(a^6+b^6+c^6+2a^3b^3+2b^3c^3+2c^3a^3)+9\geq$$

$$\geq 3(a^4+b^4+c^4)+6(a^2b^2+b^2c^2+c^2a^2)$$

$$\Leftrightarrow 2(a^3+b^3+c^3)^2\geq 3(a^4+b^4+c^4)+6(a^2b^2+b^2c^2+c^2a^2)-9 \quad (8)$$

- By AM-GM inequality and (2). We have:

$$3(a^2b^2+b^2c^2+c^2a^2)\geq 3\cdot 3\cdot\sqrt[3]{(a^2b^2)(b^2c^2)(c^2a^2)}=9\sqrt[3]{(abc)^4}\geq 9\sqrt[3]{1^4}=9$$

$$\Leftrightarrow 6(a^2b^2+b^2c^2+c^2a^2)-9\geq 3(a^2b^2+b^2c^2+c^2a^2) \quad (9)$$

Let (8),(9): $\Rightarrow 2(a^3+b^3+c^3)^2\geq 3(a^4+b^4+c^4)+3(a^2b^2+b^2c^2+c^2a^2)$

$$\Leftrightarrow 2(a^3+b^3+c^3)^2\geq 3(a^4+b^4+c^4+a^2b^2+b^2c^2+c^2a^2)$$

$$\Leftrightarrow \frac{a^4+b^4+c^4+a^2b^2+b^2c^2+c^2a^2}{2}\leq\frac{(a^3+b^3+c^3)^2}{3} \quad (10)$$

- (5), (10): $\Rightarrow \frac{a^2}{b}+\frac{b^2}{c}+\frac{c^2}{a}\leq\frac{(a^3+b^3+c^3)^2}{3}$ *(11)*

- (1), (11): $\Rightarrow \frac{a^2}{b}+\frac{b^2}{c}+\frac{c^2}{a}=\frac{(a^3+b^3+c^3)^2}{3}$ *occurs if:*

$$\begin{cases} a=b=c>0 \\ \frac{1}{\sqrt{a^3}}+\frac{1}{\sqrt{b^3}}+\frac{1}{\sqrt{c^3}}=3 \end{cases}\Leftrightarrow a=b=c=1.$$

Solution of equation is: $(a,b,c)=(1,1,1)$.

5.4 $\Delta_n=\begin{vmatrix} 1 & 2 & 3 & \cdots & n \\ 1 & 2^3 & 3^3 & \cdots & n^3 \\ 1 & 2^5 & 3^5 & \cdots & n^5 \\ \cdots & \cdots & \cdots & \cdots & \cdots \\ 1 & 2^{2n-1} & 3^{2n-1} & \cdots & n^{2n-1} \end{vmatrix}, n\geq 2$

Find:

$$\boldsymbol{\Omega = \lim_{n\to\infty}\left(\frac{\Delta_{n+1}}{n^{2n+1}\cdot\Delta_n}\right)}$$

Daniel Sitaru

Solution (Marian Ursărescu)

First, we will prove $\Delta_n = 1!\cdot 3!\cdot \ldots \cdot (2n-1)!\ (1)$ *by mathematical induction.*

$$For\ P(2): \Delta_2 = \begin{vmatrix} 1 & 2 \\ 1 & 2^3 \end{vmatrix} = 6\ true.$$

Let $P(n)$ *true and we will prove* $P(n+1)$

$$Let\ f(x) = \begin{vmatrix} 1 & 2 & \cdots & n & x \\ 1^3 & 2^3 & \cdots & n^3 & x^3 \\ \vdots & \vdots & \cdots & \vdots & \vdots \\ 1^{2n+1} & 2^{2n+1} & \cdots & n^{2n+1} & x^{2n+1} \end{vmatrix}$$

$f \in \mathbb{R}[x]$ *and grad* $f = 2n+1$ *and have* $0, \pm 1, \pm 2, \ldots, \pm n$ *roots* $\Rightarrow$

$$f(x) = a_{2n+1}(x-x_1)(x-x_2)\ldots(x-x_{2n+1}) \Rightarrow$$

$$f(x) = 1!\,3!\cdots(2n-1)!\,x(x^2-1)(x^2-2^2)\cdots(x^2-n^2) \Rightarrow$$

$$\Delta_{n+1} = 1!\,3!\cdots(2n-1)!\,(2n+1)! \Rightarrow P(n+1)\ true.$$

From (1)$\Rightarrow \Omega = \lim_{n\to\infty}\frac{\Delta_{n+1}}{n^{2n+1}\Delta_n} = \lim_{n\to\infty}\frac{(2n+1)!}{n^{2n+1}}$ *(2)*

Let $x_n = \frac{(2n+1)!}{n^{2n+1}}$

$$\lim_{n\to\infty}\frac{x_{n+1}}{x_n} = \lim_{n\to\infty}\frac{(2n+3)!}{(n+1)^{2n+3}}\cdot\frac{n^{2n+1}}{(2n+1)!} =$$

$$= \lim_{n\to\infty}\frac{(2n+3)(2n+2)}{(n+1)^2}\cdot\frac{n^{2n+1}}{(n+1)^{2n+1}} =$$

$$= \lim_{n\to\infty} 4\cdot\left(\frac{n}{n+1}\right)^{2n+1} = 4\lim_{n\to\infty}\left[\left(\frac{n}{n+1}\right)^n\right]^{\frac{2n+1}{4}} = \frac{4}{e^2} < 1\quad (3)$$

From (2)+(3)$\Rightarrow \Omega = 0.$

5.5 Let be $A \in M_5(\mathbb{R})$, invertible such that: $\det(A^2 + I_5) = 0$.

Prove that:

$$Tr\ A^* = 1 + \det A \cdot Tr\ A^{-1}$$

Marian Ursărescu

Solution (Ravi Prakash)

$$\text{As } \det(A^2 + I_5) = 0$$

$$\det[(A + iI_5)(A - iI_5)] = 0 \Rightarrow \det(A + iI_5) = 0 \text{ or } \det(A - iI_5) = 0$$

$$\Rightarrow i \text{ or } -i \text{ is an eigenvalue of } A.$$

As characteristic equation of A it has real coefficients, both $i, -i$ are eigenvalues of A. Let $\lambda_1, \lambda_2, \lambda_3$ be other eigenvalues of A.

$$Tr(A^*) = (\lambda_1 + \lambda_2 + \lambda_3)(i - i) + \lambda_1\lambda_2 + \lambda_1\lambda_3 + \lambda_2\lambda_3 + i(-i)$$

$$= \lambda_1\lambda_2 + \lambda_1\lambda_3 + \lambda_2\lambda_3 + 1$$

$$\text{Also, } \det A = \lambda_1\lambda_2\lambda_3(i)(-i) = \lambda_1\lambda_2\lambda_3$$

$$Tr(A^{-1}) = \frac{1}{\lambda_1} + \frac{1}{\lambda_2} + \frac{1}{\lambda_3} + \frac{1}{i} - \frac{1}{i} = \frac{1}{\lambda_1} + \frac{1}{\lambda_2} + \frac{1}{\lambda_3}$$

$$\det(A)\, tr(A^{-1}) = \lambda_1\lambda_2 + \lambda_1\lambda_3 + \lambda_2\lambda_3$$

$$\text{Thus, } tr(A^*) = 1 + \det(A)\, tr(A^{-1})$$

5.6 If $A, B, C \in M_n(\mathbb{C}); n \in \mathbb{N}; n \ge 2; 4A + B = 2AB;$
$9B + C = 3BC; 16C + A = 4CA$, then $ABC = CBA$.

Daniel Sitaru

Solution

$$4A + B = 2AB \Rightarrow 2A + \tfrac{1}{2}B = AB \quad (1)$$

$$(I_n - 2A)\left(I_n - \frac{1}{2}B\right) = I_n - 2A - \frac{1}{2}B + AB =$$

$$= I_n - \left(2A + \frac{1}{2}B\right) + AB \overset{(1)}{=} I_n - AB + AB = I_n$$

$$(I_n - 2A)\left(I_n - \tfrac{1}{2}B\right) = I_n \Rightarrow (I_n - 2A)^{-1} = I_n - \tfrac{1}{2}B \quad (2)$$

$$I_n=(I_n-2A)^{-1}\cdot(I_n-2A)\overset{(2)}{=}\left(I_n-\frac{1}{2}B\right)(I_n-2A)=I_n-2A-\frac{1}{2}B+BA$$

$$=I_n-\left(2A+\frac{1}{2}B\right)+BA\overset{(1)}{=}I_n-AB+BA$$

$$I_n=I_n-AB+BA\Rightarrow O_n=-AB+BA\Rightarrow AB=BA\ \ (3)$$

$$9B+C=3BC\Rightarrow 3B+\tfrac{1}{3}C=BC\ \ (4)$$

$$(I_n-3B)\left(I_n-\frac{1}{3}C\right)=I_n-3B-\frac{1}{3}C+BC=$$

$$=I_n-\left(3B+\frac{1}{3}C\right)+BC\overset{(4)}{=}I_n-BC+BC=I_n$$

$$(I_n-3B)\left(I_n-\tfrac{1}{3}C\right)=I_n\Rightarrow(I_n-3B)^{-1}=I_n-\tfrac{1}{3}C\ \ (5)$$

$$I_n=(I_n-3B)^{-1}\cdot(I_n-3B)\overset{(5)}{=}\left(I_n-\frac{1}{3}C\right)(I_n-3B)=$$

$$=I_n-3B-\frac{1}{3}C+CB=I_n-\left(3B+\frac{1}{3}C\right)+CB\overset{(4)}{=}I_n-BC+CB$$

$$I_n=I_n-BC+CB\Rightarrow O_n=-BC+CB\Rightarrow BC=CB\ \ (6)$$

$$16C+A=4CA\Rightarrow 4C+\tfrac{1}{4}A=CA\ \ (7)$$

$$(I_n-4C)\left(I_n-\frac{1}{4}A\right)=I_n-\frac{1}{4}A-4C+CA=$$

$$=I_n-\left(4C+\frac{1}{4}A\right)+CA\overset{(7)}{=}I_n-CA+CA=I_n$$

$$(I_n-4C)\left(I_n-\frac{1}{4}A\right)=I_n-4C-\frac{1}{4}A+CA=$$

$$=I_n-\left(4C+\frac{1}{4}A\right)+CA\overset{(7)}{=}I_n-CA+CA=I_n$$

$$(I_n-4C)\left(I_n-\tfrac{1}{4}A\right)=I_n\Rightarrow(I_n-4C)^{-1}=I_n-\tfrac{1}{4}A\ \ (8)$$

$$I_n=(I_n-4C)^{-1}(I_n-4C)\overset{(8)}{=}\left(I_n-\frac{1}{4}A\right)(I_n-4C)=$$

$$=I_n-4C-\frac{1}{4}A+AC=I_n-\left(4C+\frac{1}{4}A\right)+AC=I_n-CA+AC$$

$$I_n=I_n-CA+AC\Rightarrow O_n=-CA+AC\Rightarrow AC=CA\ \ (9)$$

$$ABC = A(BC) \overset{(6)}{=} A(CB) = (AC)B \overset{(9)}{=}$$

$$= (CA)B = C(AB) \overset{(3)}{=} C(BA) = CBA$$

5.7

$$\boldsymbol{x * y = x\sqrt{1 + y^2} + y\sqrt{1 + x^2}, x \circ y = xy - 5x - 5y + 30,}$$

$$\boldsymbol{G = (5, \infty)}$$

Prove that $(\mathbb{R}, *) \cong (G, \circ)$ as abelian groups.

Daniel Sitaru

Solution (Ravi Prakash)

We first show that $(\mathbb{R}, *)$, *where* $x * y = x\sqrt{1 + y^2} + y\sqrt{1 + x^2}$

is an abelian group. Clearly, $x * y \in \mathbb{R}, \forall x, y \in \mathbb{R}$

$*$ *is associative suppose* $x, y, z \in \mathbb{R}$. *Let* $x = \tan\alpha, y = \tan\beta, z = \tan\gamma$

$$-\frac{\pi}{2} < \alpha, \beta, \gamma < \frac{\pi}{2}$$

$$x * y = (\tan\alpha)\sqrt{1 + \tan^2\beta} + \tan\beta\sqrt{1 + \tan^2\alpha} = \frac{\sin\alpha + \sin\beta}{\cos\alpha\cos\beta}$$

$$(x * y) * z = \frac{\sin\alpha + \sin\beta}{\cos\alpha\cos\beta}\sqrt{1 + \tan^2\gamma} + \tan\alpha\sqrt{1 + \left(\frac{\sin\alpha + \sin\beta}{\cos\alpha\cos\beta}\right)^2}$$

But $1 + \left(\frac{\sin\alpha + \sin\beta}{\cos\alpha\cos\beta}\right)^2 = \frac{(1 - \sin^2\alpha)(1 - \sin^2\beta) + \sin^2\alpha + \sin^2\beta + 2\sin\alpha\sin\beta}{\cos^2\alpha\cos^2\beta} =$

$$\frac{(1 + \sin\alpha\sin\beta)^2}{\cos^2\alpha\cos^2\beta}$$

Thus, $(x * y) * z = \frac{\overset{\oplus}{\sin\alpha\sin\beta}}{\cos\alpha\cos\beta\cos\gamma} + \frac{\sin\gamma(1 + \sin\alpha\sin\beta)}{\cos\alpha\cos\beta\cos\gamma} =$

$$= \frac{\sin\alpha + \sin\beta + \sin\gamma + \sin\alpha\sin\beta\sin\gamma}{\cos\alpha\cos\beta\cos\gamma}$$

Similarly, $x * (y * z) = \frac{\sin\alpha + \sin\beta + \sin\gamma + \sin\alpha\sin\beta\sin\gamma}{\cos\alpha\cos\beta\cos\gamma}$

Thus, $(x * y) * z = x * (y * z); \forall x, y, z \in \mathbb{R}$

$*$ *is commutative is obvious.*

Identity Element $= 0$

$$x * 0 = x\sqrt{1+0^2} + 0\sqrt{1+x^2} = x; \forall x \in \mathbb{R}$$

Inverse Element

For each $x \in \mathbb{R}, -x \in \mathbb{R}$ *is inverse of* x.

Indeed $x * (-x) = 0$

$\therefore (\mathbb{R}, *)$ *is an abelian group. Next, we show that if* $G = (5, \infty)$*, and*

$a \circ b = ab - 5a - 5b + 30; \forall a, b \in \mathbb{G}$*, then* $(\mathbb{G}, 0)$ *is an abelian group.*

Note $a \circ b = (a-5)(b-5) + 5$

0 *is commutative and its identity element is* 6.

0 *is associative*

Let $a, b, c \in \mathbb{G}$,

$$(a \circ b) \circ c = \big((a-5)(b-5)+5\big) \circ c$$

$$= \big((a-5)(b-5)+5-5\big)(c-5)+5$$

$$= (a-5)(b-5)(c-5)+5$$

Similarly, $a \circ (b \circ c) = (a-5)(b-5)(c-5)+5$

$$\therefore (a \circ b) \circ c = a \circ (b \circ c); \forall a, b, c \in \mathbb{G}$$

Finally, if $a \in 5$*, then* $a > 5$*, and* $b = 5 + \frac{1}{a-5}$ *is inverse of* a*. Indeed,*

$a \circ b = (a-5)(b-5)+5 = (a-5)\left(\frac{1}{a-5}\right)+5 = 1+5 = 6 =$ *identity element.*

We now show that $\Phi: \mathbb{R} \to \mathbb{G}$ *defined by* $\Phi(x) = 5 + 5^{\sinh^{-1} x}$

is the required isomorphism of $\mathbb{R}$ *onto* $\mathbb{G}$

As $5^{\sinh^{-1} x} > 0, \forall x \in \mathbb{R}, \Phi(x) \in \mathbb{G}; \forall x \in \mathbb{R}$

For $x, y \in \mathbb{R}$

$$\Phi(x * y) = 5^{\sinh^{-1}\left(x\sqrt{1+y^2}+y\sqrt{1+x^2}\right)} + 5 \quad (1)$$

$$\textit{and } \Phi(x)\circ\Phi(y) = 5^{\sinh^{-1}x}\cdot 5^{\sinh^{-1}y} + 5 \quad (2)$$

$$= 5^{\sinh^{-1}x+\sinh^{-1}y} + 5$$

$$\textit{But } \sinh^{-1}x + \sinh^{-1}y = \sinh^{-1}\left(x\sqrt{1+y^2} + y\sqrt{1+x^2}\right) \quad (3)$$

∴ from (1), (2), (3): $\Phi(x*y) = \Phi(x)\circ\Phi(y)$

Thus, Φ *is a homomorphism from* $(\mathbb{R},*)$ *to* $(\mathbb{G},a)$

Φ *is one – to – one*

Let $x,y\in\mathbb{R}$ *and* $\Phi(x)=\Phi(y)$

$$\Rightarrow 5^{\sinh^{-1}x} + 5 = 5^{\sinh^{-1}y} + 5 \Rightarrow \sinh^{-1}x = \sinh^{-1}y \Rightarrow x = y$$

∴ Φ *is one – to – one*

Φ *is onto*

Let $y\in\mathbb{G} \Rightarrow y>5 \Rightarrow y-5>0$

Let $t = \log_5(y-5) \Rightarrow 5^t = y-5$

As $t\in\mathbb{R}, \exists x\in\mathbb{R}$ *such that* $\sinh^{-1}x = t$ *or take* $x = \sinh t$.

Then $\Phi(x) = 5^{\sinh^{-1}x} + 5 = 5^t + 5 = y-5+5 = y$

∴ Φ *is onto.*

Hence, $(\mathbb{R},*) \cong (\mathbb{G},0)$ *as abelian groups.*

5.8 Let be $\boldsymbol{A} = \begin{pmatrix}\mathbf{1} & \mathbf{1}\\ \mathbf{0} & \mathbf{1}\end{pmatrix}$; $\boldsymbol{B} = \begin{pmatrix}\mathbf{1} & \mathbf{0}\\ \mathbf{1} & \mathbf{1}\end{pmatrix}$**.**

Find: $\boldsymbol{\Omega} = \boldsymbol{e^A}\cdot(\boldsymbol{e^B})^{-1}$**; (**$\boldsymbol{e^A}$ **– exponential matrix)**

Daniel Sitaru

Solution (Ravi Prakash)

$$A = \begin{pmatrix}1 & 1\\ 0 & 1\end{pmatrix}, A^n = \begin{pmatrix}1 & n\\ 0 & 1\end{pmatrix}$$

$$B = \begin{pmatrix}1 & 0\\ 1 & 1\end{pmatrix}, B^n = \begin{pmatrix}1 & 0\\ n & 1\end{pmatrix}$$

$$e^A = I + A + \frac{1}{2!}A^2 + \frac{1}{3!}A^3 + \cdots$$

$$=\begin{pmatrix}1 & 0\\0 & 1\end{pmatrix}+\begin{pmatrix}1 & 1\\0 & 1\end{pmatrix}+\frac{1}{2!}\begin{pmatrix}1 & 2\\0 & 1\end{pmatrix}+\frac{1}{3!}\begin{pmatrix}1 & 3\\0 & 1\end{pmatrix}+\cdots$$

$$=\begin{pmatrix}e & e\\0 & e\end{pmatrix}=e\begin{pmatrix}1 & 1\\0 & 1\end{pmatrix}$$

Similarly, $e^B=\begin{pmatrix}e & 0\\e & e\end{pmatrix}=e\begin{pmatrix}1 & 0\\1 & 1\end{pmatrix}$

$$\Omega=e^A(e^B)^{-1}=e\begin{pmatrix}1 & 1\\0 & 1\end{pmatrix}\left(e^{-1}\begin{pmatrix}1 & 0\\-1 & 1\end{pmatrix}\right)=\begin{pmatrix}1 & 1\\0 & 1\end{pmatrix}\begin{pmatrix}1 & 0\\-1 & 1\end{pmatrix}=\begin{pmatrix}0 & 1\\-1 & 0\end{pmatrix}$$

5.9 Find $x,y,z,w\in\mathbb{R}$ such that:

$$\begin{pmatrix}\sin x & \cos y\\\tan z & \cot w\end{pmatrix}^n=\begin{pmatrix}\sin^n x & \cos^n y\\\tan^n z & \cot^n w\end{pmatrix},\forall n\in\mathbb{N}-\{0\}$$

Daniel Sitaru

Solution (Andrew Okukura)

For simplicity, we will note in $x=a,\cos y=b,\tan z=c,\cot w=d$*. Thus, the condition can be written as:* $\begin{pmatrix}a & b\\c & d\end{pmatrix}^n=\begin{pmatrix}a^n & b^n\\c^n & d^n\end{pmatrix},\forall n\in\mathbb{N}\setminus\{0\}$

For $n=2$ *we have:*

$$\begin{pmatrix}a & b\\c & d\end{pmatrix}\begin{pmatrix}a & b\\c & d\end{pmatrix}=\begin{pmatrix}a^2 & b^2\\c^2 & d^2\end{pmatrix}\Leftrightarrow\begin{pmatrix}a^2+bc & b(a+d)\\c(a+d) & d^2+bc\end{pmatrix}=\begin{pmatrix}a^2 & b^2\\c^2 & d^2\end{pmatrix}$$

$$\Rightarrow bc=0\Rightarrow b=0 \text{ or } c=0.$$

I. $b=0$*. That means the only equality is* $(ca+d)=c^2$

If $c=0$ *then the matrix* $\begin{pmatrix}0 & 0\\0 & d\end{pmatrix}$ *satisfies the identity in the hypothesis. (for any diagonal matrix* $\begin{pmatrix}0^n & 0\\0 & d^n\end{pmatrix}=\begin{pmatrix}a & 0\\0 & d\end{pmatrix}^n$*)*

If $c\neq 0\Rightarrow c=a+d$*. For* $n=3$ *we have:*

$$\begin{pmatrix}a & b\\c & d\end{pmatrix}^3=\begin{pmatrix}a^3 & b^3\\c^3 & d^3\end{pmatrix}\Leftrightarrow\begin{pmatrix}a & 0\\a+d & d\end{pmatrix}^3=\begin{pmatrix}a^3 & 0\\(a+d)^3 & d^3\end{pmatrix}\Leftrightarrow\begin{pmatrix}a^3 & 0\\ & d^3\end{pmatrix}$$

$$\Leftrightarrow\begin{pmatrix}a^3 & 0\\a(a+d)^2+d^2(a+d) & d^3\end{pmatrix}=\begin{pmatrix}a^3 & 0\\(a+d)^3 & d^3\end{pmatrix}$$

Thus, we have:

$$a(a+d)^2+d^2(a+d)=(a+d)^3|:(a+d)\Rightarrow a^2+ad+d^2$$
$$=a^2+2ad+d^2$$
$$\Rightarrow ad=0\Rightarrow a=0 \text{ or } d=0$$

If $a=b=0$ or $d=b=0$ then the matrices $\begin{pmatrix}0 & 0\\ c & d\end{pmatrix}$ and $\begin{pmatrix}a & 0\\ c & 0\end{pmatrix}$ satisfy the identity in the hypothesis.

Thus, in this case the matrices $\begin{pmatrix}a & 0\\ 0 & d\end{pmatrix}, \begin{pmatrix}0 & 0\\ c & d\end{pmatrix}$ and $\begin{pmatrix}a & 0\\ c & 0\end{pmatrix}$ satisfy the identity.

By applying the same algorithm we obtain the solutions:

$\begin{pmatrix}a & 0\\ 0 & d\end{pmatrix}$ *(duplicate),* $\begin{pmatrix}a & b\\ 0 & 0\end{pmatrix}, \begin{pmatrix}0 & b\\ 0 & d\end{pmatrix}$

Thus, the matrices $\begin{pmatrix}a & 0\\ 0 & d\end{pmatrix}, \begin{pmatrix}0 & 0\\ c & d\end{pmatrix}, \begin{pmatrix}a & 0\\ c & 0\end{pmatrix}, \begin{pmatrix}a & b\\ 0 & 0\end{pmatrix}$ and $\begin{pmatrix}0 & b\\ 0 & d\end{pmatrix}$ are the only ones which satisfy the identity above, $a,b,c,d\in\mathbb{R}$. Thus, the solutions for x,y,z,w are:

I. $x,w\in\mathbb{R}, y=k\pi+\frac{\pi}{2}\wedge z=t\pi, k,z\in\mathbb{Z}, w\neq q\pi, \forall\, q\in\mathbb{Z}$

II. $z,w\in\mathbb{R}, x=k\pi\wedge y=t\pi+\frac{\pi}{2}, k,t\in\mathbb{Z}, w\neq q\pi$ *and* $z\neq p\pi+\frac{\pi}{2}, \forall q,p\in\mathbb{Z}$

III. $x,z\in\mathbb{R}, y=t\pi+\frac{\pi}{2}, w=k\pi+\frac{\pi}{2}, k,t\in\mathbb{Z}, z\neq p\pi+\frac{\pi}{2}, \forall p\in\mathbb{Z}$

IV. $x,y\in\mathbb{R}, z=t\pi, w=k\pi+\frac{\pi}{2}, t,k\in\mathbb{Z}$

V. $y,w\in\mathbb{R}, x=k\pi, z=t\pi, t,k\in\mathbb{Z}, w\neq q\pi, \forall q\in\mathbb{Z}$

5.10 Let be $A\in M_4(\mathbb{R})$; $\det A=1$; $\det(A^2+I_n)=0$. Prove that:

$$\mathrm{Tr}(A^{-1})=\mathrm{Tr}\,A$$

Marian Ursărescu

Solution (Ravi Prakash)

As $\det(A^2+I_4)=0\Rightarrow\det[(A+iA_4)(A-iI_4)]=0$

$$\Rightarrow\det(A+iI_4)\det(A-iI_4)=0\Rightarrow\det(A+iI_4)=0 \text{ or } \det(A-iI_4)=0$$

$\Rightarrow i$ *or* $-i$ *is an eigenvalue of* A

As $A \in M_4(\mathbb{R})$*, both* $i, -i$ *are eigenvalues of* A

Let λ, μ *be other eigenvalues of* A*, then* $1 = \det(A) = i(-i)\lambda\mu = \lambda\mu$

$$\Rightarrow \lambda\mu = 1 \Rightarrow \mu = \frac{1}{\lambda}$$

$$\therefore \mathrm{Tr}(A) = i + (-i) + \lambda + \frac{1}{\lambda} = \lambda + \frac{1}{\lambda}$$

Also, $\mathrm{Tr}(A^{-1}) = \frac{1}{i} + \frac{1}{(-i)} + \frac{1}{\lambda} + \lambda = \frac{1}{\lambda} + \lambda$*.Thus,* $\mathrm{Tr}(A^{-1}) = \mathrm{Tr}(A)$

5.11 If $\boldsymbol{a, b, c > 0; n \in \mathbb{N}; n \geq 2; x_i \in \mathbb{R}; i \in \overline{1,9}}$

$$\boldsymbol{A = \begin{pmatrix} \frac{a^2}{(a+b)^2} & \frac{2ab}{(a+b)^2} & \frac{b^2}{(a+b)^2} \\ \frac{c^2}{(b+c)^2} & \frac{b^2}{(b+c)^2} & \frac{2bc}{(b+c)^2} \\ \frac{2ca}{(c+a)^2} & \frac{a^2}{(c+a)^2} & \frac{c^2}{(c+a)^2} \end{pmatrix}; A^n = \begin{pmatrix} x_1 & x_2 & x_3 \\ x_4 & x_5 & x_6 \\ x_7 & x_8 & x_9 \end{pmatrix}}$$

then find: $\boldsymbol{\Omega = \sum_{i=1}^{9} x_i}$

Daniel Sitaru

Solution:

$$A \cdot \begin{pmatrix} 1 \\ 1 \\ 1 \end{pmatrix} = \begin{pmatrix} \frac{a^2}{(a+b)^2} & \frac{2ab}{(a+b)^2} & \frac{b^2}{(a+b)^2} \\ \frac{c^2}{(b+c)^2} & \frac{b^2}{(b+c)^2} & \frac{2bc}{(b+c)^2} \\ \frac{2ca}{(c+a)^2} & \frac{a^2}{(c+a)^2} & \frac{c^2}{(c+a)^2} \end{pmatrix} \begin{pmatrix} 1 \\ 1 \\ 1 \end{pmatrix} =$$

$$= \begin{pmatrix} \frac{a^2 + 2ab + b^2}{(a+b)^2} \\ \frac{c^2 + b^2 + 2bc}{(b+c)^2} \\ \frac{2ca + a^2 + c^2}{(c+a)^2} \end{pmatrix} = \begin{pmatrix} \frac{(a+b)^2}{(a+b)^2} \\ \frac{(c+b)^2}{(c+b)^2} \\ \frac{(c+a)^2}{(c+a)^2} \end{pmatrix} = \begin{pmatrix} 1 \\ 1 \\ 1 \end{pmatrix}$$

$$A^2\cdot\begin{pmatrix}1\\1\\1\end{pmatrix}=A\cdot A\begin{pmatrix}1\\1\\1\end{pmatrix}=A\begin{pmatrix}1\\1\\1\end{pmatrix}=\begin{pmatrix}1\\1\\1\end{pmatrix}$$

By induction $:P(n): A^n\begin{pmatrix}1\\1\\1\end{pmatrix}=\begin{pmatrix}1\\1\\1\end{pmatrix}$ *(suppose true)*

$P(n+1): A^{n+1}\begin{pmatrix}1\\1\\1\end{pmatrix}=\begin{pmatrix}1\\1\\1\end{pmatrix}$ *(to prove)*

$$A^{n+1}\begin{pmatrix}1\\1\\1\end{pmatrix}=A\cdot A^n\begin{pmatrix}1\\1\\1\end{pmatrix}=A\begin{pmatrix}1\\1\\1\end{pmatrix}=\begin{pmatrix}1\\1\\1\end{pmatrix}$$

$$P(n)\to P(n+1)$$

$$A^n\begin{pmatrix}1\\1\\1\end{pmatrix}=\begin{pmatrix}1\\1\\1\end{pmatrix}\Rightarrow\begin{pmatrix}x_1&x_2&x_3\\x_4&x_5&x_6\\x_7&x_8&x_9\end{pmatrix}\begin{pmatrix}1\\1\\1\end{pmatrix}=\begin{pmatrix}1\\1\\1\end{pmatrix}$$

$$\Rightarrow\begin{pmatrix}x_1+x_2+x_3\\x_4+x_5+x_6\\x_7+x_8+x_9\end{pmatrix}=\begin{pmatrix}1\\1\\1\end{pmatrix}\Rightarrow\begin{matrix}x_1+x_2+x_3=1\\x_4+x_5+x_6=1\\x_7+x_8+x_9=1\end{matrix}$$

$$\Rightarrow\Omega=\sum_{i=1}^{9}x_i=1+1+1=3$$

5.12 Let $A\in M_3(\mathbb{R})$ invertible such that: $Tr\ A=Tr\ A^{-1}=1$. Prove that:

$$\det(A^2+A+I_3)\geq 3\det A$$

Marian Ursărescu

Solution (Ravi Prakash)

As A^{-1}, *exists,* $\det(A)\neq 0$. $\det(A^2+A+I_3)=\det\big((A-\omega I_3)(A-\omega^2 I_3)\big)$

$$=\det\big((A-\omega I_3)\big)\overline{(A-\omega I_3)}=\det(A-\omega I_3)\overline{\det(A-\omega I_3)}=|\det(A-\omega I_3)|^2\geq 0$$

∴ *If* $\det(A) < 0$, *then there is nothing to show. We assume* $\det(A) > 0$. *Let* $\det(A) = \alpha^2$, *where* $\alpha > 0$. *We have* $A^* = \det(A)\, A^{-1} = \alpha^2 A^{-1} \Rightarrow Tr\,(A^*) = \alpha^2 Tr\,(A^{-1}) = \alpha^2$. *Characteristic polynomial of* A *is:* $P(t) = \det(tI_3 - A)$

$$= t^3 - Tr\,(A)t^2 + Tr\,(A^*)t - \det(A) = (t^3 + \alpha^2 t) - (t^2 + \alpha)$$

$$= (t^2 + \alpha^2)(t - 1)$$

Now, from (1): $\det(A^2 + A + I_3) = |\det(A - \omega I_3)|^2 = |\det(\omega I_3 - A)|^2$

$$= |(\omega - 1)||\omega^2 + \alpha^2|$$

But $|\omega - 1| = \left|-\frac{3}{2} + \frac{\sqrt{3}}{2}i\right| = 3$ *and* $|\omega^2 + \alpha^2| = \left(-\frac{1}{2} + \alpha\right)^2 + \left(\frac{\sqrt{3}}{2}\right)^2$

$$= \alpha^4 - \alpha^2 + 1 \geq \alpha^2$$

Thus, $\det(A^2 + A + I_3) \geq 3\alpha^2 = 3\det(A)$

5.13 Let be $G = \{a + b\sqrt[3]{5} + c\sqrt[3]{25} | a, b, c \in \mathbb{Q}\}$

Prove that if $x \in G$ then, $x^{2019} \in G$.

Daniel Sitaru

Solution

Let be $x = a + b\sqrt[3]{5} + c\sqrt[3]{25}; a, b \in \mathbb{Q}$

$$y = d + e\sqrt[3]{5} + f\sqrt[3]{25}; d, e, f \in \mathbb{Q}$$

$$xy = (a + b\sqrt[3]{5} + c\sqrt[3]{25})(d + e\sqrt[3]{5} + f\sqrt[3]{25}) =$$

$$= ad + ae\sqrt[3]{5} + af\sqrt[3]{25} + bd\sqrt[3]{5} +$$

$$+be\sqrt[3]{25} + 5bf + dc\sqrt[3]{25} + 5ce + 5fc\sqrt[3]{5} =$$

$$= ad + 5bf + 5ce + (ae + bd + 5fc)\sqrt[3]{5} +$$

$$+(af + be + dc)\sqrt[3]{25} \in G$$

because: $ad + 5bf + 5ce; ae + bd + 5fc; af + be + dc \in \mathbb{Q}$

$$x, y \in G \Rightarrow xy \in G \quad (1)$$

$$x \in G \overset{(1)}{\Rightarrow} x \cdot x \in G \overset{(2)}{\Rightarrow} x^2 \cdot x \in G \Rightarrow x^n \in G; n \in \mathbb{N}^*$$

Inductively. For $n = 2019 \Rightarrow x^{2019} \in G$.

5.14 If $A\in M_3(\mathbb{R}); Tr(A^2)=0; \det=1$ then:

$$\det(A^2+A+I_3)\ge(Tr\ A)^3$$

Marian Ursărescu

Solution (Florentin Vișescu)

$$\text{If } Tr(A^2)=0 \text{ and } \det A=1$$

$$P_A(x)=x^3-Tr\ A\ x^2+\frac{(Tr\ A)^2}{2}x-1$$

$$\det(A^2+A+I_3)=\det(A-\varepsilon I_3)(A-\bar{\varepsilon}I_3)=$$

$$=P_A(\varepsilon)\cdot P_A(\bar{\varepsilon})=\left(\varepsilon^3-Tr\ A\ \varepsilon^2+\frac{(Tr\ A)^2}{2}\varepsilon-1\right)\cdot\left(\bar{\varepsilon}^3-Tr\ A\bar{\varepsilon}^2+\frac{(Tr\ A)^2}{2}\bar{\varepsilon}-1\right)$$

$$=(Tr\ A)^2\varepsilon\cdot\bar{\varepsilon}\left(\varepsilon-\frac{Tr\ A}{2}\right)\left(\bar{\varepsilon}-\frac{Tr\ A}{2}\right)$$

$$=(Tr\ A)^2\cdot\left(\varepsilon\bar{\varepsilon}-\varepsilon\frac{Tr\ A}{2}-\bar{\varepsilon}\frac{Tr\ A}{2}+\frac{(Tr\ A)^2}{4}\right)$$

$$=(Tr\ A)^2\cdot\left(1-\frac{Tr\ A}{2}(\varepsilon+\bar{\varepsilon})+\frac{(Tr\ A)^2}{4}\right)=(Tr\ A)^2\cdot\left(1+\frac{Tr\ A}{2}+\frac{(Tr\ A)^2}{4}\right)\ge(Tr\ A)^3$$

$$1+\frac{Tr\ A}{2}+\frac{(Tr\ A)^2}{4}\ge Tr\ A,\qquad 4+2\,Tr\ A+(Tr\ A)^2\ge 4Tr\ A$$

$$(Tr\ A)^2-2Tr\ A+4\ge 0,\qquad (Tr\ A-1)^2+3\ge 0$$

5.15 Find $A=\begin{pmatrix}\cos t & -\sin t\\ \sin t & \cos t\end{pmatrix}, t\in\mathbb{R}$, such that:

$$A^4-4A^3+6A^2-4A+I_2=O_2$$

Daniel Sitaru

Solution (Ravi Prakash)

$$A=\begin{pmatrix}\cos t & -\sin t\\ \sin t & \cos t\end{pmatrix}$$

$$\det(A)=\cos^2 t+\sin^2 t=1\Rightarrow A^{-1}\text{ exists and } A^{-1}=\begin{pmatrix}\cos t & \sin t\\ -\sin t & \cos t\end{pmatrix}$$

$$\text{Let } B=A+A^{-1}=\begin{pmatrix}2\cos t & 0\\ 0 & 2\cos t\end{pmatrix}$$

$$\text{Given equation is } A^4-4A^3+6A^2-4A+I_2=O_2$$

$$\Rightarrow A^2-4A+6I_2-4A^{-1}+A^{-2}=O_2 \Rightarrow A^2+A^{-2}-4(A+A^{-1})+6I_2=O_2$$

$$\Rightarrow (A+A^{-1})^2-4(A+A^{-1})+4I_2=O_2 \Rightarrow (A+A^{-1}-2I_2)^2=0$$

$$\Rightarrow (B-2I_2)^2=O_2$$

$$\therefore \begin{pmatrix} 2\cos t-2 & 0 \\ 0 & 2\cos t-2 \end{pmatrix}^2=O_2 \Rightarrow 4\begin{pmatrix} (\cot t-1)^2 & 0 \\ 0 & (\cos t-1)^2 \end{pmatrix}=0$$

$$\Rightarrow (\cot t-1)^2=0 \Rightarrow \cot t=1, \sin t=0 \therefore A=\begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}=I_2$$

5.16 Find $a,b,c,d\in\mathbb{R}$ such that:

$$\begin{cases} \begin{pmatrix} a & b \\ -b & a \end{pmatrix}+\begin{pmatrix} c & d \\ -d & c \end{pmatrix}=\begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix} \\ \begin{pmatrix} a & b \\ -b & a \end{pmatrix}^3+\begin{pmatrix} c & d \\ -d & c \end{pmatrix}^3=\begin{pmatrix} 2 & 0 \\ 0 & 2 \end{pmatrix} \end{cases}$$

Daniel Sitaru

Solution (Daniel Văcaru)

We note: $\begin{pmatrix} a & b \\ -b & a \end{pmatrix}=X$ *and* $\begin{pmatrix} c & d \\ -d & c \end{pmatrix}=Y$.

Our system could be written as $\begin{cases} X+Y=I_2 \\ X^3+Y^3=2I_2 \end{cases}$

Let's observe that $XY=YX$, *then* $b=0; d=0$.

It follows $a_{1,2}=\frac{3\pm\sqrt{21}}{6} \Rightarrow c_{1,2}=1-\frac{3\pm\sqrt{21}}{6}=\frac{3\mp\sqrt{21}}{6}$

We obtain:

$$X=\begin{pmatrix} \frac{3+\sqrt{21}}{6} & 0 \\ 0 & \frac{3+\sqrt{21}}{6} \end{pmatrix}, Y=\begin{pmatrix} \frac{3-\sqrt{21}}{6} & 0 \\ 0 & \frac{3-\sqrt{21}}{6} \end{pmatrix} \text{ and}$$

$$X=\begin{pmatrix} \frac{3-\sqrt{21}}{6} & 0 \\ 0 & \frac{3-\sqrt{21}}{6} \end{pmatrix}, Y=\begin{pmatrix} \frac{3+\sqrt{21}}{6} & 0 \\ 0 & \frac{3+\sqrt{21}}{6} \end{pmatrix}$$

5.17

$$\boldsymbol{A}=\begin{pmatrix}\sin^2 a & \cos^2 a\cdot\sin^2 b & \cos^2 a\cdot\cos^2 b\\ \cos^2 b\cdot\sin^2 c & \sin^2 b & \cos^2 b\cdot\cos^2 c\\ \cos^2 c\cdot\sin^2 a & \cos^2 c\cdot\cos^2 a & \sin^2 c\end{pmatrix}$$

$$A^{2019}=\begin{pmatrix}x_1 & x_2 & x_3\\ x_4 & x_5 & x_6\\ x_7 & x_8 & x_9\end{pmatrix}$$

If $a,b,c\in\mathbb{R}$ then find:

$$\Omega=\sum_{i=1}^{9}x_i$$

Proposed by Daniel Sitaru – Romania

Solution (Ravi Prakash)

$$Let\ x=\begin{pmatrix}1\\1\\1\end{pmatrix}$$

$$AX=\begin{pmatrix}\sin^2 a+\cos^2 a\sin^2 b+\cos^2 a\cos^2 b\\ \cos^2 b\sin^2 c+\sin^2 b+\cos^2 b\cos^2 c\\ \cos^2 c\sin^2 a+\cos^2 c\sin^2 a+\sin^2 c\end{pmatrix}=\begin{pmatrix}1\\1\\1\end{pmatrix}=x$$

Thus $A^2(x)=A(AX)=AX=x$. Continuing in this way, we get

$$A^{2019}X=X\Rightarrow\begin{pmatrix}x_1 & x_2 & x_3\\ x_4 & x_5 & x_6\\ x_7 & x_8 & x_9\end{pmatrix}\begin{pmatrix}1\\1\\1\end{pmatrix}=\begin{pmatrix}1\\1\\1\end{pmatrix}\Rightarrow\begin{pmatrix}x_1+x_2+x_3\\ x_4+x_5+x_6\\ x_7+x_8+x_9\end{pmatrix}=\begin{pmatrix}1\\1\\1\end{pmatrix}$$

$$\therefore\sum_{k=1}^{9}x_k=3$$

5.18 Let be $A\in M_5(\mathbb{R})$ such that $AA^T=I_5$ and $Tr\ A=Tr\ A^2=0$.

Find A^{2020}.

Marian Ursărescu

Solution

The matrix A being orthogonal one $\Rightarrow|\lambda_1|=|\lambda_2|=|\lambda_3|=|\lambda_4|=|\lambda_5|=1$

$$P_A(x)=x^5-ax^4+bx^3-cx^2+dx-\det A=0 \quad (1)$$

$$a=Tr\ A=0 \quad (2)$$

$$B=Tr\ A^*=\frac{1}{2}((Tr\ A)^2-Tr\ A^2)=0 \quad (3)$$

$$d=\sum\lambda_1\lambda_2\lambda_3\lambda_4=\sum\lambda_1\lambda_2\lambda_3\lambda_4\lambda_5\left(\frac{1}{\lambda_1}+\frac{1}{\lambda_2}+\frac{1}{\lambda_3}+\frac{1}{\lambda_4}+\frac{1}{\lambda_5}\right)$$

$$=\det A\cdot\sum\left(\overline{\lambda_1}+\overline{\lambda_2}+\overline{\lambda_3}+\overline{\lambda_4}+\overline{\lambda_5}\right)=\det A\cdot\overline{\sum(\lambda_1+\lambda_2+\lambda_3+\lambda_4+\lambda_5)}$$

$$=\det A\cdot\overline{Tr\ A}=0\ (A)$$

$$C=\sum\lambda_1\lambda_2\lambda_3=\lambda_1\lambda_2\lambda_3\lambda_4\lambda_5\sum\frac{1}{\lambda_1\lambda_2}=\det A\cdot\sum\overline{\lambda_1\lambda_2}$$

$$=\det A\cdot\left(\overline{Tr\ A^2}\right)=0 \quad (5)$$

From (1)+(2)+(3)+(4)+(5)$\Rightarrow P_A(x)=x^5-\det A\Rightarrow$

$$\left.\begin{array}{c}A^5=\det AI_5\\ But\ \det(AA^T)=\det I_5\Rightarrow(\det A)^2=1\Rightarrow\det A=\pm1\end{array}\right\}\Rightarrow$$

$$\Rightarrow A^5=\pm I_5\Rightarrow(A^5)^{404}=I_5^{404}\Rightarrow A^{2020}=I_5$$

5.19 If $A,B\in M_4(\Omega)$;

$$AB=\begin{pmatrix}p&p&p&p\\0&-p&-p&-p\\0&0&p&p\\0&0&0&-p\end{pmatrix};p\in\mathbb{C},p\neq0;\ \Omega_1=BA;$$

$\Omega_2=(BA)^{-1}$ **then find:**

$$\Omega=\Omega_1^2+\left(p^2\Omega_2^{-1}\right)^2$$

Marian Ursărescu

Solution (Florentin Vişescu)

$$P_{AB}(x)=det(xI_4-AB)=det\begin{pmatrix}x-p&-p&-p&-p\\0&x+p&p&p\\0&0&x-p&-p\\0&0&0&x+p\end{pmatrix}=$$

$$=(x-p)^2(x+p)^2=x^4-2p^2x^2+p^4\Rightarrow$$

$$P_{AB}(x)=x^4-2p^2x^2+p^4\Leftrightarrow(BA)^4-2p^2(BA)^2+p^4I_4=O_4\Leftrightarrow$$
$$(BA)^4+p^4I_4=2p^2(BA)^2/_sB^{-1}/_dA^{-1}\Rightarrow(AB)^3+p^4B^{-1}A^{-1}$$
$$=2p^2AB/_sA^{-1}/_dB^{-1}\Rightarrow$$
$$(BA)^2+p^4(BA)^{-1}(BA)^{-1}=2p^2,\ (BA)^2+(p^2(BA)^{-1})^2=2p^2I_4$$
$$\Omega=\Omega_1^2+(p^2\Omega_2^{-1})^2=2p^2I_4$$

5.20 If $A\in M_2(\mathbb{R}); Tr\ A=\det A=1$ then:

$$\det(A^2+3A+3I_2)\geq 5Tr(A^{-1})+3$$

Marian Ursărescu

Solution (Florentin Vișescu)

$$Tr\ A=\det A=1\Rightarrow A^2-A+I_2=O_2$$
$$A^2=A-I_2;A^2-A=-I_2\Rightarrow A-A^2=I_2\Rightarrow$$
$$A(I_2-A)=I_2\Rightarrow A^{-1}=I_2-A$$

(*) $5Tr(A^{-1})+3=5\big(Tr(I_2-A)\big)+3=5(2-1)+3=5+3=8$

(**) $\det(A^2+3A+3I_2)=\det(A-I_2+3A+3I_2)$

$$=\det(4A+2I_2)=\det\big(2(2A+I_2)\big)=4\det(2A+I_2)$$

So, we have to prove that $\det(2A+I_2)\geq 2$

Let be $A=\begin{pmatrix}a & b\\ c & d\end{pmatrix}\in M_2(\mathbb{R}); Tr\ A=1\Rightarrow a+d=1$

$$\Rightarrow d=1-a\Rightarrow A=\begin{pmatrix}a & b\\ c & 1-a\end{pmatrix}\det A=1\Rightarrow a-a^2-bc=1$$
$$\Rightarrow a-a^2-1=bc$$

If $b=0\Rightarrow a^2-a+1=0\Rightarrow a\in\mathbb{C}$ *(False). So,* $b\neq 0\Rightarrow c=\frac{a-a^2-1}{b}\Rightarrow$

$$A=\begin{pmatrix}a & b\\ \frac{a-a^2-1}{b} & 1-a\end{pmatrix}\Rightarrow 2A+I_2=\begin{pmatrix}2a+1 & 2b\\ \frac{2a-2a^2-2}{b} & 3-2a\end{pmatrix}$$
$$\det(2A+I_2)=(2a+1)(3-2a)-4a+4a^2+4=$$
$$=6a-4a^2+3-2a-4a+4a^2+4=7$$

5.21 If $A \in M_4(\mathbb{Q}), \det\left((1-i)A+\sqrt{2}I_4\right)=0$ then:

$$\det(A+xI_4) \geq 2x^2, x \in \mathbb{R}$$

Marian Ursărescu

Solution (Florentin Vișescu)

In these conditions: $\det\left((1-i)A+\sqrt{2}I_4\right)=0 \Leftrightarrow \det\left(A+\frac{\sqrt{2}+i\sqrt{2}}{2}I_4\right)=0$

We denote $\theta=\frac{\sqrt{2}+i\sqrt{2}}{2} \Rightarrow \theta$ *root for*

$\det(A+xI_4)$. *But* $\det(A+xI_4)$ *has rational coefficients* $\Rightarrow \bar{\theta}$ *root of it*

As $\det(A+xI_4)$ *has the grade IV, we obtain:*

$$\det(A+xI_4)=(x-\theta)(x-\bar{\theta})(x^2+ax+b)$$

$$=\left(x^2-\sqrt{2}x+1\right)(x^2+ax+b)=$$

$$=x^4+ax^3+bx^2-\sqrt{2}x^3-a\sqrt{2}x^2-\sqrt{2}xb+x^2+ax+b$$

$$=x^4+x^3\left(a-\sqrt{2}\right)+x^2\left(b-a\sqrt{2}+1\right)+x\left(a-b\sqrt{2}\right)+b \in \mathbb{Q}[x] \Rightarrow b \in \mathbb{Q}$$

$$a-\sqrt{2} \in \mathbb{Q} \Rightarrow a-\sqrt{2}=q \in \mathbb{Q} \Rightarrow a=q+\sqrt{2}$$

$$b-a\sqrt{2}+1 \in \mathbb{Q} \Rightarrow b-a\sqrt{2} \in \mathbb{Q} \Rightarrow a-b\sqrt{2} \in \mathbb{Q} \Rightarrow$$

$$\Rightarrow b-\left(q+\sqrt{2}\right)\sqrt{2} \in \mathbb{Q} \Rightarrow b-q\sqrt{2}-2 \in \mathbb{Q}$$

$$\Rightarrow b-q\sqrt{2} \in \mathbb{Q} \Rightarrow b-q\sqrt{2}=p \in \mathbb{Q}$$

or $-q\sqrt{2}=p-b$ *or* $q\sqrt{2}=b-p \in \mathbb{Q}$

If $q \neq 0 \Rightarrow \sqrt{2}=\frac{b-p}{q}$ *(False) so,* $q=0 \Rightarrow a=\sqrt{2} \Rightarrow \sqrt{2}-b\sqrt{2} \in \mathbb{Q}$

$$\sqrt{2}(1-b) \in \mathbb{Q} \Rightarrow 1-b=0 \Rightarrow b=1$$

Then:

$$\det(A+xI_4)=\left(x^2-\sqrt{2}x+1\right)\left(x^2+\sqrt{2}x+1\right)$$

$$\det(A+xI_4)=(x^2+1)^2-2x^2=x^4+2x^2+1-2x^2=x^4+1$$

$$x^4+1 \geq 2x^2$$

$x^4-2x^2+1 \geq 0$ $(x^2-1)^2 \geq 0$ *(True)*

5.22 ***If*** $A\in M_6(\mathbb{R})$ **such that:**

$$det(A^4+pA^2+p^2I_6)=det(A^2+qI_6)=0, p,q\in\mathbb{R} \text{ then find:}$$

$$\Omega=det(A)$$

Marian Ursărescu

Solution

Let: $f(x)=x^4+px^2+p^2; g(x)=x^2+q; f,g\in\mathbb{R}[x]$

P_A *–the characteristic polynomial of matrix* A.

We must show that: $(f,P_A)\neq 1$.

Suppose that: $(f,P_A)=1\Rightarrow \exists u,v\in\mathbb{R}[x]$ *such that:* $f(x)u(x)+$

$P_A(x)v(x)=1\Rightarrow$

$f(A)u(A)+P_A(A)v(A)=I_6\Rightarrow I_6=O_6$ *contradiction!*

Similary $(f,g)\neq 1\Rightarrow f/P_A$ *and* g/P_A, *for degree* $P_A=6$

But $(f,g)=1\Rightarrow P_A(x)=f(x)\cdot g(x)\Rightarrow$

$det(A)=P_A(0)=f(0)\cdot g(0)=p^2q$

5.23 $X,Y\in M_2(\mathbb{R}), X^{19}+X^{17}=Y^{21}+Y^{19}=\begin{pmatrix}1 & -1\\ -1 & 1\end{pmatrix}$

Find:

$$\Omega=\lim_{n\to\infty}\sqrt[n]{\frac{Tr\,(X^{n+1})}{Tr\,(Y^{n+2})}}$$

Daniel Sitaru

Solution (Marian Ursărescu)

$$X\in M_2(\mathbb{R}); x^{19}+x^{17}=\begin{pmatrix}1 & -1\\ -1 & 1\end{pmatrix}=A$$

$$X^{19}+X^{17}=A\Rightarrow \left.\begin{matrix}X^{20}+X^{18}=AX\\ X^{20}+X^{18}=XA\end{matrix}\right\}\Rightarrow AX=XA$$

$$X=\begin{pmatrix} a & b\\ c & d\end{pmatrix}\Rightarrow \left.\begin{matrix} AX=\begin{pmatrix} a & b\\ c & d\end{pmatrix}\begin{pmatrix} 1 & -1\\ -1 & 1\end{pmatrix}=\begin{pmatrix} a-b & -a+b\\ c-d & -c+d\end{pmatrix}\\ XA=\begin{pmatrix} 1 & -1\\ -1 & 1\end{pmatrix}\begin{pmatrix} a & b\\ c & d\end{pmatrix}=\begin{pmatrix} a-c & b-d\\ -a+c & -b+d\end{pmatrix}\end{matrix}\right\}\Rightarrow$$

$$b=c; a=d\Rightarrow X=\begin{pmatrix} a & b\\ b & a\end{pmatrix}\Rightarrow X^n$$

$$=\begin{pmatrix} \frac{(a+b)^n+(a-b)^n}{2} & \frac{(a+b)^n-(a-b)^n}{2}\\ \frac{(a+b)^n-(a-b)^n}{2} & \frac{(a+b)^n+(a-b)^n}{2}\end{pmatrix}$$

$$\Rightarrow X^{19}+X^{17}=A\Rightarrow\begin{cases} \frac{(a+b)^{19}+(a-b)^{19}}{2}+\frac{(a+b)^{17}+(a-b)^{17}}{2}=1\\ \frac{(a+b)^{19}-(a-b)^{19}}{2}+\frac{(a+b)^{17}-(a-b)^{17}}{2}=-1\end{cases}$$

$$\Rightarrow (a+b)^{19}+(a+b)^{17}=0\Rightarrow a+b=0 \textit{ unique solution } b=-a$$

$$\Rightarrow (a-b)^{19}+(a-b)^{17}=2\Rightarrow a-b=1\Rightarrow a=\frac{1}{2}, b=\frac{1}{2}$$

$$X=\begin{pmatrix} \frac{1}{2} & -\frac{1}{2}\\ -\frac{1}{2} & \frac{1}{2}\end{pmatrix}. \textit{ The same for } Y.$$

5.24 If $A\in M_2(\mathbb{R})$ such that $det(A^4+4I_2)=0$.Prove that:

$$(\boldsymbol{detA})^2=(\boldsymbol{trA})^2$$

Marian Ursărescu

Solution

$$det(A^4+4I_2)=det(A^4+4A^2+4I_2^2-4A^2)=det[(A^2+2I_2)^2-4A^2]$$

$$=det(A^2+2A+2I_2)\cdot det(A^2-2A+2I_2)=0$$

$$det(A^2+2A+2I_2)=0 \textit{ or } det(A^2-2A+2I_2)=0$$

We have:

$$\textit{If } det(A^2+2A+2I_2)=0\Leftrightarrow det(A+I_2+iI_2)\cdot det(A+I_2-iI_2)=0\Leftrightarrow$$

$$det(A+I_2+iI_2)=0 \textit{ or } det(A+I_2-iI_2)=0 \quad (1)$$

$$\textit{Let: } p(x)=det(A+I_2+xI_2)=det(A+I_2)+a_ix+x^2, a_i\in\mathbb{R}$$

$$\stackrel{(1)}{\Rightarrow} p(\pm i)=0 \Rightarrow det(A+I_2)-1\pm a_i=0 \Rightarrow det(A+I_2)=1$$

Let: $A=\begin{pmatrix} a & b \\ c & d \end{pmatrix}; A+I_2=\begin{pmatrix} a+1 & b \\ c & d+1 \end{pmatrix} \Rightarrow$

$$det(A+I_2)=ad+a+d+1=det(A)+tr(A)+1=1$$

$$\Rightarrow det(A)+tr(A)=0$$

Analogous, from $det(A^2-2A+2I_2)=0 \Rightarrow det(A)-tr(A)=0 \quad (3)$

From (2)+(3) we have: $(detA)^2=(trA)^2$

5.25 If $A \in M_n(\mathbb{R}); A^3=2A^2+7A+4I_n$ then find:

$$\Omega=\det(A^2-3A+3I_n)$$

Marian Ursărescu

Solution

$$A^3-2A^2-7A-4I_n=O_n$$

Let: $f(x)=x^3-2x^2-7x-4=(x+1)^2(x-4) \Rightarrow f(A)=O_n$

Let be m_A *–the minimal polynomial* $m_A/f \Rightarrow m_A(x)=(x+1)^{k_1}(x-4)^{k_2}$,

from Frobenius theorem, result $p_A(x)=(x+1)^{p_1}(x-4)^{p_2}, p_1+p_2=n; (1)$

$$p_A(x)=det(xI_n-A)=(-1)^n det(A-xI_n); (2)$$

Let be the equation: $x^2-3x+3=0$ *with roots* $x_{1,2}=\frac{-3\pm i\sqrt{3}}{2}=1\pm\frac{1\pm i\sqrt{3}}{2}$

Where the equation: $x^2-x+1=0$ *have the roots* $\frac{1\pm i\sqrt{3}}{2}$.

$$\alpha=1+\varepsilon; \varepsilon^2-\varepsilon+1=0, \varepsilon^3=-1, \varepsilon^6=1, \varepsilon+\bar{\varepsilon}=1, \varepsilon\cdot\bar{\varepsilon}=1.$$

$$det(A^2-3A+3I_n)=det(A-\alpha I_n)\cdot det(A-\bar{\alpha}I_n); (3)$$

From (2) result: $p_A(\alpha)=(-1)^n det(A-\alpha I_n);\ p_A(\bar{\alpha})=(-1)^n det(A-\bar{\alpha}I_n)$

$$p_A(\alpha)\cdot p_A(\bar{\alpha})=det(A-\alpha I_n)\cdot det(A-\bar{\alpha}I_n)=det(A^2-3A+3I_n); (4)$$

$$p_A(\alpha)\cdot p_A(\bar{\alpha})=(\alpha+1)^{p_1}(\alpha-4)^{p_2}(\bar{\alpha}+1)^{p_1}(\bar{\alpha}-4)^{p_2}=$$

$$=\big((\varepsilon+2)(\bar{\varepsilon}+2)\big)^{p_1}\big((\varepsilon-3)(\bar{\varepsilon}-3)\big)^{p_2}$$

$$=(1+2(\varepsilon+\bar{\varepsilon})+4)^{p_1}(1-3(\varepsilon+\bar{\varepsilon})+9)^{p_2}=$$

$$=7^{p_1}\cdot 7^{p_2}=7^{p_1+p_2}=7^n \Rightarrow \Omega=det(A^2-3A+3I_n)=7^n.$$

5.26 $\boldsymbol{A, B \in M_{2019}(\mathbb{R}), p \in \mathbb{R} - \{0\}}$,

$\boldsymbol{I_{2019} + 2p(A+B) + 2p^2(A^2+B^2) = O_{2019}}$.

Find: $\boldsymbol{\Omega = det(AB - BA)}$

Marian Ursărescu

Solution (Florentin Vişescu)

$$I_{2019} + 2p(A+B) + 2p^2(A^2+B^2) = O_{2019} \Leftrightarrow$$

$$2I_{2019} + 4p(A+B) + 4p^2(A^2+B^2) = O_{2019} \Leftrightarrow$$

$$(I_{2019} + 2pA)^2 + (I_{2019} + 2pB)^2 = O_{2019}$$

Let: $C = I_{2019} + 2pA$ and $D = I_{2019} + 2pB, C, D \in M_{2019}(\mathbb{R})$ then

$$C^2 + D^2 = O_{2019}$$

$$(C+iD)(C-iD) = C^2 - iCD + iDC + D^2 = i(DC - CD) \Rightarrow$$

$$det(C+iD)(C-iD) = det(C+iD)det(C-iD)$$

$$= det(C+iD)\overline{det(C+\imath D)} \geq 0$$

$$\Rightarrow det[i(DC-CD)] \geq 0 \Rightarrow i^{2019}det(DC-CD) \geq 0$$

$$\Rightarrow i^3 det(DC-CD) \geq 0 \Rightarrow det(DC-CD) = 0$$

$$DC - CD = (I_{2019} + 2pB)(I_{2019} + 2pA) - (I_{2019} + 2pA)(I_{2019} + 2pB)$$

$$= I_{2019} + 2pA + 2pB + 4p^2BA - I_{2019} - 2pB - 2pA - 4p^2AB = 4p^2(BA - AB)$$

$$det(DC - CD) = det[4p^2(BA - AB)] = (-4p^2)^{2019}det(BA - AB) \Rightarrow$$

$$\Omega = det(AB - BA) = 0$$

5.27 Prove that:

$$\boldsymbol{\begin{vmatrix} sinxsiny & siny & 3sinx & 3 \\ sinx & 1 & sinxcosy & cosy \\ 2siny & sinysinx & 6 & 3cosx \\ 2 & cosx & 2cosy & cosxcosy \end{vmatrix} \neq 0; \forall x, y \in \mathbb{R}}$$

Daniel Sitaru

Solution (Ravi Prakash)

$$Let: \Delta=\begin{vmatrix} sinxsiny & siny & 3sinx & 3 \\ sinx & 1 & sinxcosy & cosy \\ 2siny & sinysinx & 6 & 3cosx \\ 2 & cosx & 2cosy & cosxcosy \end{vmatrix}$$

$$Using\ c_1 \to c_1-(sinx)c_2\ and\ c_3 \to c_3-(sinx)c_4$$

$$\Delta=\begin{vmatrix} 0 & siny & 0 & 2 \\ 0 & 1 & 0 & cosy \\ siny(2-sinxcosx) & sinysinx & 3(2-cos^2x) & 3cosx \\ 2-sinxcosx & cosx & cosy(2-cos^2x & cosxcosy \end{vmatrix}=$$

$$=(2-sinxcosx)(2-cos^2x)\Delta_1$$

$$\Delta_1=\begin{vmatrix} 0 & siny & 0 & 3 \\ 0 & 1 & 0 & cosy \\ siny & sinycosx & 3 & 3cosx \\ 1 & cosx & cosy & cosxcosy \end{vmatrix}$$

$$Using\ c_2 \to c_2-(cosx)c_1\ and\ c_4 \to c_4-(cosx)c_3$$

$$\Delta_1=\begin{vmatrix} 0 & siny & 0 & 3 \\ 0 & 1 & 0 & cosy \\ siny & 0 & 3 & 0 \\ 1 & 0 & cosy & 0 \end{vmatrix}$$

$$Using\ r_3 \to r_3-(cosx)r_4, \text{ we get:}$$

$$\Delta_1=\begin{vmatrix} 0 & siny & 0 & 3 \\ 0 & 1 & 0 & cosy \\ 0 & 0 & 3-sinycosy & 0 \\ 1 & 0 & cosy & 0 \end{vmatrix} \overset{\substack{Expanding \\ c_1}}{=}$$

$$-\begin{vmatrix} siny & 0 & 3 \\ 1 & 0 & cosy \\ 0 & 3-sinycosy & 0 \end{vmatrix}=$$

$$\overset{\substack{Expanding \\ r_3}}{=} \quad (3-sinycosy)\begin{vmatrix} siny & 3 \\ 1 & cosy \end{vmatrix}=$$

$$=-(2-sinxcosx)(2-cos^2x)(3-sinycosy)^2 \neq 0$$

5.28

$$A=\{x/x>0,x^{\sqrt{x}}=4\cdot 2^{\sqrt{x}}\};B=\{y/y>0,y^{\sqrt{y}}=27\cdot 9^{\sqrt[3]{y}}\}$$

$C=\{z/z>0,z^{\sqrt{z}}=256\cdot 64^{\sqrt[4]{z}}\}$. **Find the set Ω such that:**

$$A\Delta\Omega\Delta B=C,(X\Delta Y=(X/Y)\cup(Y/X)$$

Daniel Sitaru

Solution(Khanh Hung Vu)

We have $x^{\sqrt{x}}=4\cdot 2^{\sqrt{x}}\Rightarrow log\left(x^{\sqrt{x}}\right)=log\left(4\cdot 2^{\sqrt{x}}\right)\Rightarrow$

$$\sqrt{x}log\ x=log4+\sqrt{x}log2\Rightarrow\sqrt{x}log\ x-log4-\sqrt{x}log2=0;(1)$$

Put: $f_1(x)=\sqrt{x}log\ x-log4-\sqrt{x}log2;\ f'_1(x)=\frac{1}{2\sqrt{x}}\left(log\left(\frac{x}{2}\right)+2\right)$

$$f'_1(x)=0\Leftrightarrow log\left(\frac{x}{2}\right)+2=0\Leftrightarrow x=\frac{2}{e^2}$$

We have:

$$\lim_{x\to 0^+}(f_1(x))=\lim_{x\to 0^+}(\sqrt{x}log\ x-log4-\sqrt{x}log2)$$

$$=\lim_{x\to 0^+}\left(\frac{log\ x}{\frac{1}{\sqrt{x}}}\right)-log4\overset{L'H}{=}$$

$$=\lim_{x\to 0^+}\left(\frac{\frac{1}{x}}{\frac{-1}{x\sqrt{x}}}\right)-log4=\lim_{x\to 0^+}(-2\sqrt{x})-log4=-log4$$

$$\lim_{x\to\infty}(f_1(x))=\lim_{x\to\infty}(\sqrt{x}log\ x-log4-\sqrt{x}log2)$$

$$=\lim_{x\to 0^+}\left(\sqrt{x}log\left(\frac{x}{2}\right)-log4\right)=+\infty$$

So, the equation (1) has only one root, which is $x=4$, then $A=\{4\}$.

Since $4\notin C$ and since $z^{\sqrt{z}}=16$ and $256\cdot 64^{\sqrt[4]{z}}\notin\mathbb{N}$, so not have exist the set Ω such that

$$A\Delta\Omega\Delta B=C$$

MISCELLANEOUS INEQUALITIES

6.1 If $a,b,c\geq 1, a,b,c\in\mathbb{N}$ then:

$$\boldsymbol{a\left(\binom{2b}{b}+\binom{2c}{c}\right)+b\left(\binom{2c}{c}+\binom{2a}{a}\right)+c\left(\binom{2a}{a}+\binom{2b}{b}\right)}$$
$$\boldsymbol{\geq 2\left(4^{\sqrt{ab}}+4^{\sqrt{bc}}+4^{\sqrt{ca}}\right)}$$

Daniel Sitaru

Solution

$$a\binom{2b}{b}+b\binom{2a}{a}\overset{AM-GM}{\geq} 2\sqrt{a^b\binom{2b}{b}\cdot\binom{2a}{a}}=$$

$$=2\sqrt{ab\cdot\sum_{k=0}^{a}\binom{a}{k}^2\cdot\sum_{k=0}^{b}\binom{b}{k}^2}\overset{CBS}{\geq} 2\sqrt{ab\cdot\frac{\left(\sum_{k=0}^{a}\binom{a}{k}\right)^2}{a}\cdot\frac{\left(\sum_{k=0}^{b}\binom{b}{k}\right)^2}{b}}=$$

$$=2\cdot\sum_{k=0}^{a}\binom{a}{k}\cdot\sum_{k=0}^{b}\binom{b}{k}=2\cdot 2^a\cdot 2^b=2\cdot 2^{a+b}\overset{AM-GM}{\geq} 2\cdot 2^{2\sqrt{ab}}=2\cdot 4^{\sqrt{ab}}$$

$$a\binom{2b}{b}+b\binom{2a}{a}\geq 2\cdot 4^{\sqrt{ab}}\quad (1)$$

$$b\binom{2c}{c}+c\binom{2b}{b}\geq 2\cdot 4^{\sqrt{bc}}\quad (2)$$

$$c\binom{2a}{a}+a\binom{2c}{c}\geq 2\cdot 4^{\sqrt{ca}}\quad (3)$$

By adding (1); (2); (3):

$$a\left(\binom{2b}{b}+\binom{2c}{c}\right)+b\left(\binom{2c}{c}+\binom{2a}{a}\right)+c\left(\binom{2a}{a}+\binom{2b}{b}\right)$$
$$\geq 2\left(4^{\sqrt{ab}}+4^{\sqrt{bc}}+4^{\sqrt{ca}}\right)$$

Equality holds for $a=b=c=1$.

6.2 If $a,b,c \geq 1$; $a,b,c \in \mathbb{N}$ then:

$$\frac{1}{\binom{2a}{a}^2}\sum_{k=0}^{a}\binom{a}{k}^3+\frac{1}{\binom{2b}{b}^2}\sum_{k=0}^{b}\binom{b}{k}^3+\frac{1}{\binom{2c}{c}^2}\sum_{k=0}^{c}\binom{c}{k}^3 \geq \frac{9}{2^a+2^b+2^c}$$

Daniel Sitaru

Solution

$$\binom{a}{0}^3+\binom{a}{1}^3+\dots+\binom{a}{a}^3=\frac{\binom{a}{0}^4}{\binom{a}{0}}+\frac{\binom{a}{1}^4}{\binom{a}{1}}+\dots+\frac{\binom{a}{a}^4}{\binom{a}{a}}\geq$$

$$\overset{BERGSTROM}{\geq}\frac{\left(\binom{a}{0}^2+\binom{a}{1}^2+\dots+\binom{a}{a}^2\right)^2}{\binom{a}{0}+\binom{a}{1}+\dots+\binom{a}{a}}=\frac{1}{2^a}\binom{2a}{a}^2$$

$$\frac{1}{\binom{2a}{a}^2}\sum_{k=0}^{a}\binom{a}{k}^3\geq\frac{1}{2^a}\quad (1)$$

$$\frac{1}{\binom{2b}{b}^2}\sum_{k=0}^{b}\binom{b}{k}^3\geq\frac{1}{2^b}\quad (2)$$

$$\frac{1}{\binom{2c}{c}^2}\sum_{k=0}^{c}\binom{c}{k}^3\geq\frac{1}{2^c}\quad (3)$$

By adding (1); (2); (3):

$$\frac{1}{\binom{2a}{a}^2}\sum_{k=0}^{a}\binom{a}{k}^3+\frac{1}{\binom{2b}{b}^2}\sum_{k=0}^{b}\binom{b}{k}^3+\frac{1}{\binom{2c}{c}^2}\sum_{k=0}^{c}\binom{c}{k}^3\geq$$

$$\geq\frac{1}{2^a}+\frac{1}{2^b}+\frac{1}{2^c}\overset{BERGSTROM}{\geq}\frac{(1+1+1)^2}{2^a+2^b+2^c}=\frac{9}{2^a+2^b+2^c}$$

Equality holds for $a=b=c=1$.

6.3 If $a,b,c>0$ then:

$$\frac{abc(a+b)(b+c)(c+a)}{8}\leq\left(\frac{a+b+c}{3}\right)^6$$

Daniel Sitaru

Solution

$$abc \overset{AM-GM}{\leq} \left(\frac{a+b+c}{3}\right)^3 \quad (1)$$

$$(a+b)(b+c)(c+a) \overset{AM-GM}{\leq} \left(\frac{a+b+b+c+c+a}{3}\right)^3 =$$

$$= \left(\frac{2(a+b+c)}{3}\right)^3 = \frac{8(a+b+c)^3}{3^3} = 8\left(\frac{a+b+c}{3}\right)^3$$

$$(a+b)(b+c)(c+a) \leq 8\left(\frac{a+b+c}{3}\right)^3 \quad (2)$$

By multiplying (2); (3):

$$abc(a+b)(b+c)(c+a) \leq 8\left(\frac{a+b+c}{3}\right)^6$$

$$\frac{abc(a+b)(b+c)(c+a)}{8} \leq \left(\frac{a+b+c}{3}\right)^6$$

6.4 If $x, y \geq 0, 0 \leq z \leq 1$ then:

$$4\sqrt{xy} \leq 2(1-z)\left(\sqrt{x}+\sqrt{y}\right)\sqrt[4]{xy} + z\left(\sqrt{x}+\sqrt{y}\right)^2 \leq 2(x+y)$$

Daniel Sitaru-Romania

Solution (Tran Hong)

$$\left(\sqrt{x}+\sqrt{y}\right)^2 \overset{CBS}{\leq} \left(\sqrt{1^2+1^2}+\sqrt{\sqrt{x}^2+\sqrt{y}^2}\right)^2 = 2(x+y)$$

$$\left(\sqrt{x}+\sqrt{y}\right)\sqrt[4]{xy} \overset{Am-Gm}{\leq} \left(\sqrt{x}+\sqrt{y}\right)\frac{\sqrt{x}+\sqrt{y}}{2} = \frac{\left(\sqrt{x}+\sqrt{y}\right)^2}{2} \leq \sqrt{x}^2+\sqrt{y}^2$$

$$= x+y$$

$$\overset{0\leq z\leq 1}{\Longrightarrow} \Omega = 2(1-z)\left(\sqrt{x}+\sqrt{y}\right)\sqrt[4]{xy} + z\left(\sqrt{x}+\sqrt{y}\right)^2 \leq (2-2z+2z)(x+y)$$

$$= 2(x+y)$$

$$\left(\sqrt{x}+\sqrt{y}\right)^2 \overset{Am-Gm}{\geq} 4\sqrt{xy} \Rightarrow \sqrt{x}+\sqrt{y} \geq 2\sqrt[4]{xy} \Rightarrow \left(\sqrt{x}+\sqrt{y}\right)\sqrt[4]{xy} \geq 2\sqrt{xy}$$

$$\underset{0\leq z\leq 1}{\Longrightarrow} \Omega \geq \left(4z+4(1-z)\right)\sqrt{xy} = 4\sqrt{xy}$$

6.5 If $0 < a \le b < \frac{\pi}{6}$ then:

$$\sin(5\sqrt{ab})\cdot\sin\left(\frac{12ab}{a+b}\right)\ge\sin(6\sqrt{ab})\cdot\sin\left(\frac{10ab}{a+b}\right)$$

Daniel Sitaru

Solution (Tran Hong)

$$\sin(5\sqrt{ab})\cdot\sin\left(\frac{12ab}{a+b}\right)=\frac{1}{2}\left[\cos\left(5\sqrt{ab}-\frac{12ab}{a+b}\right)-\cos\left(5\sqrt{ab}+\frac{12ab}{a+b}\right)\right]$$

$$\sin(6\sqrt{ab})\cdot\sin\left(\frac{10ab}{a+b}\right)=\frac{1}{2}\left[\cos\left(6\sqrt{ab}-\frac{10ab}{a+b}\right)-\cos\left(6\sqrt{ab}+\frac{10ab}{a+b}\right)\right]$$

Must show that:

$$\cos\left(5\sqrt{ab}-\frac{12ab}{a+b}\right)+\cos\left(6\sqrt{ab}+\frac{10ab}{a+b}\right)$$

$$\ge\cos\left(6\sqrt{ab}-\frac{10ab}{a+b}\right)+\cos\left(5\sqrt{ab}+\frac{12ab}{a+b}\right)$$

$$\Leftrightarrow 2\left\{\cos\left[\frac{ab}{a+b}-\frac{11\sqrt{ab}}{2}\right]\cdot\cos\left[\frac{\sqrt{ab}}{2}+\frac{11ab}{a+b}\right]\right\}\ge 2\left\{\cos\left[\frac{11\sqrt{ab}}{2}+\frac{ab}{a+b}\right]\cdot\cos\left[\frac{\sqrt{ab}}{2}-\frac{11ab}{a+b}\right]\right\}\quad (*)$$

$$\cos\left[\frac{11ab}{a+b}+\frac{\sqrt{ab}}{2}\right]\overset{(1)}{\ge}\cos\left[\frac{11\sqrt{ab}}{2}+\frac{ab}{a+b}\right]$$

$$\Leftrightarrow\frac{11\sqrt{ab}}{2}+\frac{ab}{a+b}\ge\frac{11ab}{a+b}+\frac{\sqrt{ab}}{2}$$

$$\Leftrightarrow 5\sqrt{ab}\ge 10\cdot\frac{ab}{a+b}\Leftrightarrow a+b\ge 2\sqrt{ab}\quad \textit{(true)}$$

$$\cos\left(\frac{ab}{a+b}-\frac{11\sqrt{ab}}{2}\right)\overset{(2)}{\ge}\cos\left(\frac{\sqrt{ab}}{2}-\frac{11ab}{a+b}\right)$$

$$\Leftrightarrow\frac{\sqrt{ab}}{2}-\frac{11ab}{a+b}\ge\frac{ab}{a+b}-\frac{11\sqrt{ab}}{2}\Leftrightarrow 6\sqrt{ab}\ge\frac{12ab}{a+b}\Leftrightarrow a+b\ge 2\sqrt{ab}\quad \textit{(true)}$$

From (1) and (2) we have: (*) true.

6.6 If $0 < a_1 \le a_2 \le \ldots \le a_n, n > 0$, then prove:

$$\frac{a_1 \cdot a_n}{n}\left(\sum_{k=1}^{n}\frac{1}{a_k}\right)^2 \le \sum_{k=1}^{n}\left(\frac{a_1+a_n}{a_k}-1\right)$$

Florică Anastase

Solution(Adrian Popa)

$$Let: \sum_{k=1}^{n}\frac{1}{a_k}=x$$

$$\frac{a_1a_n}{n}\cdot x^2 \overset{?}{\le} (a_1+a_n)x-n$$

$$a_1a_nx^2-(a_1+a_n)nx+n^2 \overset{?}{\le} 0; \quad (1)$$

$$\Delta = n^2(a_n-a_1)^2>0$$

$$x_{1,2}=\frac{n(a_1+a_n)\pm n(a_n-a_1)}{2a_1a_n} \Rightarrow x_1=\frac{n}{a_1}, x_2=\frac{n}{a_n}$$

X	$x_1=\frac{n}{a_1}$ $\quad x_2=\frac{n}{a_n}$
E(x)	$+++++0----0+++++$

$$But: x=\frac{1}{a_1}+\frac{1}{a_2}+\cdots+\frac{1}{a_n}$$

$$a_1 \le a_2 \le \cdots \le a_n \Rightarrow \frac{1}{a_1} \ge \frac{1}{a_2} \ge \cdots \ge \frac{1}{a_n} \Rightarrow \frac{n}{a_n} \le \sum_{k=1}^{n}\frac{1}{a_k} \le \frac{n}{a_1} \Rightarrow (1)-it's$$

true.So:

$$\frac{a_1 \cdot a_n}{n}\left(\sum_{k=1}^{n}\frac{1}{a_k}\right)^2 \le \sum_{k=1}^{n}\left(\frac{a_1+a_n}{a_k}-1\right)$$

6.7 If $a,b,c>0$ then:

$$\frac{(2a+b+c)(a+2b+c)(a+b+2c)}{(a+b)(b+c)(c+a)} \ge 8$$

Daniel Sitaru

Solution

$$\text{Let be } f:(0,1)\to\mathbb{R};\ f(x)=\log\left(\frac{1+x}{1-x}\right)$$

$$f(x)=\log(1+x)-\log(1-x)$$

$$f'(x)=\frac{1}{1+x}+\frac{1}{1-x}$$

$$f''(x)=\frac{-1}{(1+x)^2}+\frac{1}{(1-x)^2}=\frac{-1+2x-x^2+1+2x+x^2}{(1-x^2)^2}$$

$$f''(x)=\frac{4x}{(1-x^2)^2}>0;(\forall)x\in(0,1)\Rightarrow f\ \text{convexe}$$

By Jensen's inequality:

$$f\left(\frac{\frac{a}{a+b+c}+\frac{b}{a+b+c}+\frac{c}{a+b+c}}{3}\right)\le\frac{1}{3}\sum_{cyc}f\left(\frac{a}{a+b+c}\right)$$

$$f\left(\frac{1}{3}\right)\le\frac{1}{3}\sum_{cyc}\log\left(\frac{1+\frac{a}{a+b+c}}{1-\frac{a}{a+b+c}}\right),3f\left(\frac{1}{3}\right)\le\log\left(\prod_{cyc}\left(\frac{2a+b+c}{b+c}\right)\right)$$

$$3\log\left(\frac{1+\frac{1}{3}}{1-\frac{1}{3}}\right)\le\log\left(\prod_{cyc}\left(\frac{2a+b+c}{b+c}\right)\right),\log 8\le\log\left(\prod_{cyc}\left(\frac{2a+b+c}{b+c}\right)\right)$$

$$\prod_{cyc}\left(\frac{2a+b+c}{b+c}\right)\ge 8$$

6.8

$$\boldsymbol{\Omega_1=\frac{1}{(1-a^2)^7}+\frac{1}{(1-b^2)^7}+\frac{1}{(1-c^2)^7},}$$

$$\boldsymbol{\Omega_2=\frac{1}{(1-x^2)^7}+\frac{1}{(1-y^2)^7}+\frac{1}{(1-z^2)^7}}$$

$$\boldsymbol{\Omega_3=\frac{1}{(1-ax)^7}+\frac{1}{(1-by)^7}+\frac{1}{(1-cz)^7},a,b,c,x,y,z\in(-1,1)}$$

Prove that: $\boldsymbol{\Omega_1\Omega_2\ge\Omega_3^2}$

Daniel Sitaru

Solution (Soumava Chakraborty)

$$(1-a^2)(1-x^2) \le (1-ax)^2 \Leftrightarrow 1-a^2-x^2+a^2x^2 \le 1+a^2x^2-2ax$$

$$\Leftrightarrow (a-x)^2 \ge 0 \to true \Rightarrow \frac{1}{\sqrt{(1-a^2)(1-x^2)}} \overset{(1)}{\ge} \frac{1}{|1-a^x|}$$

$$Similarly, \frac{1}{\sqrt{(1-b^2)(1-y^2)}} \overset{(2)}{\ge} \frac{1}{|1-by|} \; and \; \frac{1}{\sqrt{(1-c^2)(1-z^2)}} \overset{(3)}{\ge} \frac{1}{|1-cz|}$$

$$Let \; \frac{1}{\sqrt{1-a^2}} = A, \frac{1}{\sqrt{1-b^2}} = B, \frac{1}{\sqrt{1-c^2}} = C$$

$$\frac{1}{\sqrt{1-x^2}} = X, \frac{1}{\sqrt{1-y^2}} = Y, \frac{1}{\sqrt{1-z^2}} = Z$$

$$\therefore \Omega_1\Omega_2 = \left(\sum A^{14}\right)\left(\sum x^{14}\right) \overset{CBS}{\ge} \left\{\sum (AX)^7\right\}^2 \ge \left(\sum \left|\frac{1}{1-ax}\right|^7\right)^2$$

$$\text{(using (1), (2), (3))} = \left(\sum \left|\left(\frac{1}{1-ax}\right)^7\right|\right)^2 \ge \left(\sum \left(\frac{1}{1-ax}\right)^7\right)^2 = \Omega_3^2 \; (\because |u| \ge u)$$

6.9 If $a, b > 0$ then ($\emptyset - golden\ ratio$):

$$\left((1+\emptyset)^{\sqrt{ab}} + \sqrt{\emptyset^{a+b}}\right)\left((1+\pi)^{\sqrt{ab}} + \sqrt{\pi^{a+b}}\right)$$

$$\le \left(\emptyset^{\sqrt{ab}} + \sqrt{(1+\emptyset)^{a+b}}\right)\left(\pi^{\sqrt{ab}} + \sqrt{(1+\pi)^{a+b}}\right)$$

Daniel Sitaru

Solution (Florentin Vişescu)

Let be the functions: $f, g: [x; x+1] \to \mathbb{R}; x \ge 1; f(t) = t^{\sqrt{ab}}; g(t) = t^{\frac{a+b}{2}}$

$f'(t) = \sqrt{ab} \cdot t^{\sqrt{ab}-1}; g'(t) = \frac{a+b}{2} \cdot t^{\frac{a+b}{2}-1} \overset{Cauchy-T}{\Longrightarrow} \exists c \in (x; x+1)$ *such that:*

$$\frac{f'(c)}{g'(c)} = \frac{f(x+1)-f(x)}{g(x+1)-g(x)} \Leftrightarrow \frac{\sqrt{ab}\cdot c^{\sqrt{ab}-1}}{\frac{a+b}{2}\cdot t^{\frac{a+b}{2}-1}} = \frac{(x+1)^{\sqrt{ab}} - x^{\sqrt{ab}}}{(x+1)^{\frac{a+b}{2}} - x^{\frac{a+b}{2}}} \Leftrightarrow$$

$$\frac{(x+1)^{\sqrt{ab}} - x^{\sqrt{ab}}}{(x+1)^{\frac{a+b}{2}} - x^{\frac{a+b}{2}}} = \frac{\sqrt{ab}\cdot c^{\sqrt{ab}-\frac{a+b}{2}}}{\frac{a+b}{2}}; (1)$$

$$c \in (x; x+1); x \geq 1 \; and \; \sqrt{ab} - \frac{a+b}{2} \leq 0 \Rightarrow c^{\sqrt{ab}-\frac{a+b}{2}} \leq 1; (2)$$

From (1),(2) we have:

$$\frac{(x+1)^{\sqrt{ab}} - x^{\sqrt{ab}}}{(x+1)^{\frac{a+b}{2}} - x^{\frac{a+b}{2}}} \leq 1 \Leftrightarrow (x+1)^{\sqrt{ab}} - x^{\sqrt{ab}} \leq (x+1)^{\frac{a+b}{2}} - x^{\frac{a+b}{2}} \Leftrightarrow$$

$$x^{\frac{a+b}{2}} + (x+1)^{\sqrt{ab}} \leq x^{\sqrt{ab}} + (x+1)^{\frac{a+b}{2}}$$

$$\text{For } x = \emptyset \Rightarrow \emptyset^{\frac{a+b}{2}} + (\emptyset+1)^{\sqrt{ab}} \leq \emptyset^{\sqrt{ab}} + (\emptyset+1)^{\frac{a+b}{2}}$$

$$\text{For } x = \pi \Rightarrow \pi^{\frac{a+b}{2}} + (\pi+1)^{\sqrt{ab}} \leq \pi^{\sqrt{ab}} + (\pi+1)^{\frac{a+b}{2}}$$

$$\left((1+\emptyset)^{\sqrt{ab}} + \sqrt{\emptyset^{a+b}}\right)\left((1+\pi)^{\sqrt{ab}} + \sqrt{\pi^{a+b}}\right) \leq \left(\emptyset^{\sqrt{ab}} + \sqrt{(1+\emptyset)^{a+b}}\right)\left(\pi^{\sqrt{ab}} + \sqrt{(1+\pi)^{a+b}}\right)$$

6.10 If $x, y, z > 0$ then:

$$\frac{(1+\sqrt{x})(1+\sqrt[3]{y})(1+\sqrt[6]{z})}{\sqrt{1+x} \cdot \sqrt[3]{1+y} \cdot \sqrt[6]{1+z}} \geq 4$$

Daniel Sitaru

Solution (Sanong Huayrerai)

For $x, y, z > 0$ we give $x = a^2; y = b^3; z = c^6$

Hence $\frac{(1+\sqrt{x})(1+\sqrt[3]{y})(1+\sqrt[6]{z})}{\sqrt{1+x} \cdot \sqrt[3]{1+y} \cdot \sqrt[6]{1+z}} \geq 4$

$$(1+a)(1+b)(1+c) \leq 4\sqrt{1+a^2} \cdot \sqrt[3]{1+b^3} \cdot \sqrt[6]{1+c^6}$$

$$= 4\sqrt[6]{(1+a^2)^3(1+b^3)^2(1+c^6)}$$

$$(1+a)^6(1+b)^6(1+c)^6 \leq 4^6 \cdot (1+a^2)^3(1+b^3)^2(1+c^6)$$

true, because

$$(1+a)^6 \leq 2^3(1+a^2)^3$$

$$(1+b)^6 \leq 2^4(1+b^3)^2$$

$$(1+c)^6 \leq 2^5(1+c^6)$$

6.11 If $a,b,c,d,e,f>0, a+b+c=3, d+e+f=9$ then:

$$\frac{a^a\cdot b^b\cdot c^c\cdot d^d\cdot e^e\cdot f^f}{3(a+d)^{a+d}(b+e)^{b+e}(c+f)^{c+f}}\ge\left(\frac{3}{8}\right)^8$$

Daniel Sitaru

Solution (Tran Hong)

For $a,b,c,d,e,f>0$ we have:

$$a^a\cdot\left(\frac{d}{3}\right)^{\frac{d}{3}}\cdot\left(\frac{d}{3}\right)^{\frac{d}{3}}\cdot\left(\frac{d}{3}\right)^{\frac{d}{3}}\ge\left(\frac{a+\frac{d}{3}+\frac{d}{3}+\frac{d}{3}}{4}\right)^{a+\frac{d}{3}+\frac{d}{3}+\frac{d}{3}}=\frac{(a+d)^{a+d}}{4^{a+d}}\Rightarrow$$

$$(a+d)^{a+d}\le 4^{a+d}\cdot a^a\cdot\frac{d^d}{3^d}\text{ and analogs}$$

$$(b+e)^{b+e}\le 4^{b+e}\cdot b^b\cdot\frac{e^e}{3^e},\quad (c+f)^{c+f}\le 4^{c+f}\cdot c^c\cdot\frac{f^f}{3^f}$$

$$(a+d)^{a+d}\cdot(b+e)^{b+e}\cdot(c+f)^{c+f}$$

$$\le 4^{a+b+c+d+e+f}\cdot a^a\cdot b^b\cdot c^c\cdot d^d\cdot e^e\cdot f^f\cdot\frac{1}{3^{d+e+f}}$$

$$=4^{12}\cdot a^a\cdot b^b\cdot c^c\cdot d^d\cdot e^e\cdot f^f\cdot\frac{1}{3^9}\Leftrightarrow\frac{a^a\cdot b^b\cdot c^c\cdot d^d\cdot e^e\cdot f^f}{(a+d)^{a+d}(b+e)^{b+e}(c+f)^{c+f}}\ge\frac{3^9}{4^{12}}$$

$$\Leftrightarrow\frac{a^a\cdot b^b\cdot c^c\cdot d^d\cdot e^e\cdot f^f}{3(a+d)^{a+d}(b+e)^{b+e}(c+f)^{c+f}}\ge\left(\frac{3}{8}\right)^8$$

Proved.Equality for $a=b=c=1$ and $d=e=f=3$

6.12 If $a,b,c,d>0, a+b+c+d=12$ then:

$$\frac{ab}{12(a+b)}+\frac{abc}{8(ab+bc+ca)}+\frac{abcd}{6(abc+bcd+cda+dab)}<1$$

Daniel Sitaru

Solution (Tran Hong)

$$\frac{1}{a+b}\overset{CBS}{\le}\frac{1}{4}\left(\frac{1}{a}+\frac{1}{b}\right)\Rightarrow\frac{ab}{12(a+b)}\le\frac{1}{48}(a+b)\quad(1)$$

$$\frac{1}{ab+bc+ca}\overset{CBS}{\leq}\frac{1}{9}\left(\frac{1}{ab}+\frac{1}{bc}+\frac{1}{ca}\right)\Rightarrow\frac{abc}{8(ab+bc+ca)}\leq\frac{1}{72}(a+b+c)\ \ (2)$$

$$\frac{1}{abc+bcd+cda+dab}\leq\frac{1}{16}\left(\frac{1}{abc}+\frac{1}{bcd}+\frac{1}{cda}+\frac{1}{dab}\right)\Rightarrow$$

$$\frac{abcd}{6(abc+bcd+cda+dab)}\leq\frac{1}{96}(a+b+c+d)\quad(3)\overset{(1)+(2)+(3)}{\Longrightarrow}$$

$$LHS\leq\frac{1}{48}(a+b)+\frac{1}{72}(a+b+c)+\frac{1}{96}(a+b+c+d)$$

$$=\frac{13}{288}a+\frac{13}{288}b+\frac{13}{288}c+\frac{1}{96}d<\frac{a+b+c+d}{12}=1$$

6.13 If $a,b,c>0, abc=1$ then:

$$\frac{(a+b)\sqrt{c}}{2}\left(1-\frac{(a+b)\sqrt{c}}{2}\right)\leq\frac{2}{(a+b)\sqrt{c}}\left(\frac{2}{(a+b)\sqrt{c}}-1\right)$$

Daniel Sitaru

Solution (Adrian Popa)

$$(a+b)\sqrt{c}\overset{Am-Gm}{\geq}2\sqrt{ab}\cdot\sqrt{c}=2\sqrt{abc}=2\rightarrow\frac{(a+b)\sqrt{c}}{2}\geq1$$

Let: $\frac{(a+b)\sqrt{c}}{2}=t$

We must show that: $t(1-t)\leq\frac{1}{t}\left(\frac{1}{t}-1\right);\forall t\geq1$

$$t-t^2\leq\frac{1}{t^2}-\frac{1}{t}\leftrightarrow t+\frac{1}{t}\leq t^2+\frac{1}{t^2}\leftrightarrow t+\frac{1}{t}\leq\left(t+\frac{1}{t}\right)^2-2$$

Let: $t+\frac{1}{t}=u; u\geq2$

We must show that: $u\leq u^2-2\leftrightarrow u^2-u-2\geq0$

$$\Delta=9, u_1=-1; u_2=2$$

u	$-\infty$ -1 2 $+\infty$
u^2-u-2	$++++0--0++++$

So, $u^2-u-2\geq0;\ \forall u\geq2$ *then*

$$\frac{(a+b)\sqrt{c}}{2}\left(1-\frac{(a+b)\sqrt{c}}{2}\right)\le\frac{2}{(a+b)\sqrt{c}}\left(\frac{2}{(a+b)\sqrt{c}}-1\right)$$

6.14 If $0\le a\le 2, b\ge 0, 0\le c\le 1, a+b+c=3, x, y, z>0$, $xyz=1$ then:

$$\frac{1}{ax^2+bx+c}+\frac{1}{ay^2+by+c}+\frac{1}{az^2+bz+c}\ge 1$$

Rahim Shahbazov

Solution (Tran Hong)

Because: $xyz=1$, *let*: $x=\frac{vt}{u^2}, y=\frac{ut}{v^2}, z=\frac{uv}{t^2}$

Inequality becomes as: $\sum\frac{u^4}{av^2t^2+bvtu^2+cu^4}\ge 1;\ \ (1)$

$$\sum\frac{u^4}{av^2t^2+bvtu^2+cu^4}\overset{C-B-S}{\ge}$$

$$\ge\frac{(u^2+v^2+t^2)^2}{c(u^4+v^4+t^4)+a(v^2t^2+u^2t^2+u^2v^2)+buvt(u+v+t)}$$

So, we must show that:

$$(u^2+v^2+t^2)^2\ge$$

$$\ge c(u^4+v^4+t^4)+a(v^2t^2+u^2t^2+u^2v^2)+buvt(u+v+t);$$

$$(1-c)(u^4+v^4+t^4)+(2-a)(v^2t^2+u^2t^2+u^2v^2)\ge$$

$$\ge buvt(u+v+t)=(3-a-c)uvt(u+v+t)$$

Which is clearly true because: $1-c\ge 0; 2-a\ge 0$

$$u^4+v^4+t^4\ge v^2t^2+u^2t^2+u^2v^2\ge uvt(u+v+t)$$

So, $(1-c)(u^4+v^4+t^4)+(2-a)(v^2t^2+u^2t^2+u^2v^2)\ge$

$$(1-c-2-a)uvt(u+v++t)=(3-a-c)uvt(u+v+t)$$

6.15 If $a, b, c, x, y, z>0; a^x\cdot b^y\cdot c^z=1$ then:

$$a^{x^2}\cdot b^{y^2}\cdot c^{z^2}\cdot(a+b)^{2xy}\cdot(b+c)^{2yz}\cdot(c+a)^{2zx}\ge 4^{xy+yz+zx}$$

Daniel Sitaru

Solution

$$\frac{a+b}{2} \geq \sqrt{ab} \Rightarrow a+b \geq 2\sqrt{ab} \quad \textit{(AM-GM)}$$

$$(a+b)^{2xy} \geq \left(2\sqrt{ab}\right)^{2xy} = 4^{xy} \cdot (ab)^{xy} \quad (1)$$

$$(b+c)^{2yz} \geq 4^{yz} \cdot (bc)^{yz} \quad (2)$$

$$(c+a)^{2zx} \geq 4^{zx} \cdot (ca)^{zx} \quad (3)$$

By multiplying (1); (2); (3):

$$(a+b)^{2xy} \cdot (b+c)^{2yz} \cdot (c+a)^{2zx} \geq 4^{xy+yz+zx} \cdot (ab)^{xy} \cdot (bc)^{yz} \cdot (ca)^{zx}$$

$$a^{x^2} \cdot b^{y^2} \cdot c^{z^2} \cdot (a+b)^{2xy} \cdot (b+c)^{2yz} \cdot (c+a)^{2zx} \geq$$

$$\geq 4^{xy+yz+zx} \cdot (ab)^{xy} \cdot (bc)^{yz} \cdot (ca)^{zx} \cdot a^{x^2} \cdot b^{y^2} \cdot c^{z^2} =$$

$$= 4^{xy+yz+zx} \cdot (a^x)^{x+y+z} \cdot (b^y)^{x+y+z} \cdot (c^z)^{x+y+z} =$$

$$= 4^{xy+yz+zx} \cdot (a^x \cdot b^y \cdot c^z)^{x+y+z} = 4^{xy+yz+zx} \cdot 1^{x+y+z} = 4^{xy+yz+zx}$$

Equality holds for $a = b = c = x = y = z = 1$

6.16 If $a, b > 0$ then:

$$\frac{a+b}{\sqrt{ab}} + \frac{4ab}{(a+b)^2} \leq \frac{4\sqrt{ab}}{a+b} + \frac{(a+b)^2}{4ab}$$

Daniel Sitaru

Solution

Inequality can be written: $\frac{a+b}{\sqrt{ab}} + \frac{4}{\left(\frac{a+b}{\sqrt{ab}}\right)^2} \leq \frac{4}{\frac{a+b}{\sqrt{ab}}} + \frac{\left(\frac{a+b}{\sqrt{ab}}\right)^2}{4}$

Denote $x = \frac{a+b}{\sqrt{ab}}$

By AM-GM; $\frac{a+b}{2} \geq \sqrt{ab} \Rightarrow \frac{a+b}{\sqrt{ab}} \geq 2 \Rightarrow x \geq 2$

$$x + \frac{4}{x^2} \leq \frac{4}{x} + \frac{x^2}{4}, 4x^3 + 16 \leq 16x + x^4$$

$$x^4 - 4x^3 + 16x - 16 \geq 0$$

$$x^4 - 2x^3 - 2x^3 + 4x^2 - 4x^2 + 8x + 8x - 16 \geq 0$$

$$x^3(x-2)-2x^2(x-2)-4x(x-2)+8(x-2)\geq 0$$

$$(x-2)(x^3-2x^2-4x+8)\geq 0, (x-2)\left(x^2(x-2)-4(x-2)\right)\geq 0$$

$$(x-2)^2(x^2-4)\geq 0, (x-2)^3(x+2)\geq 0$$

Which is true because

$$x\geq 2\Rightarrow x-2\geq 0; x+2>0$$

6.17 If $x, y\in\left(0,\frac{\pi}{2}\right)$ then:

$$\frac{3}{\sin x}-\frac{1}{2\sin y\cos x}+\frac{1}{2\cos x\cos y}<\frac{6}{\sin 2x\sin 2y\cos x}$$

Daniel Sitaru

Solution (Tran Hong)

$$\frac{3}{\sin x}-\frac{1}{2\sin y\cos x}+\frac{1}{2\cos x\cos y}<\frac{6}{\sin 2x\sin 2y\cos x}$$

$$\Leftrightarrow\frac{6\cos^2 x\sin 2y+\sin 2x\,(\sin y-\cos y)}{\sin 2x\sin 2y\cos x}<\frac{6}{\sin 2x\sin 2y\cos x}$$

Because: $0<x,y<\frac{\pi}{2}\Rightarrow\sin 2x, \sin 2y, \cos x>0$. *We need to prove:*

$$6\cos^2 x\sin 2y+\sin 2x\,(\sin y-\cos y)<6$$

$$\Leftrightarrow 3(1+\cos 2x)\sin 2y+\sin 2x\,(\sin y-\cos y)<6$$

$$\Leftrightarrow 3\cos 2x\sin 2y+\sin 2x\,(\sin y-\cos y)<3\quad (*)$$

We have:

$$LHS_{(*)}\leq 3\left|\cos 2x\,|\sin 2y|+|\sin 2x||\sin y-\cos y|\right|\overset{(BCS)}{\leq}\sqrt{9\sin^2 2y+1-\sin 2y}\overset{(1)}{\leq}3$$

(1)$\Leftrightarrow 9\sin^2 2y-\sin 2y<8\quad (t=\sin 2y, 0<t<1)$

$\Leftrightarrow 9t^2-t-8<0\Leftrightarrow 9(t-1)\left(t+\frac{8}{9}\right)<0$; *(True)* $\Rightarrow$ *(1) true* $\Rightarrow$ *(*) true.*

6.18 If $0<a<b<\frac{\pi}{2}$ then:

$$\frac{e^{\sin b}-e^{\sin a}}{\sin b-\sin a}>1+\frac{\sin(a+b)}{2}$$

Nguyen Van Nho

Solution (Ravi Prakash)

For $0 < \alpha < \beta < 1$

$$\frac{e^{\beta} - e^{\alpha}}{\beta - \alpha} = \frac{1}{\beta - \alpha}\left[(\beta - \alpha) + \frac{1}{2!}(\beta^2 - \alpha^2) + \frac{1}{3!}(\beta^3 - \alpha^3) + \cdots\right]$$

$$= 1 + \frac{1}{2!}(\beta + \alpha) + \frac{1}{3!}(\beta^2 + \beta\alpha + \alpha^2) + \cdots > 1 + \frac{1}{2}(\beta + \alpha)$$

$$\Rightarrow \frac{e^{\sin b} - e^{\sin a}}{\sin b - \sin a} > 1 + \frac{1}{2}(\sin b + \sin a)$$

$$\geq 1 + \frac{1}{2}(\sin b \cos a + \sin a \cos b) = 1 + \frac{1}{2}\sin(b + a)$$

6.19 If $0 < x, y, z < \frac{\pi}{6}$ then:

$$(\sin^2 x)^{\sin\left(\frac{y+z}{2}\right)\cos\left(\frac{y-z}{2}\right)} + (\sin^2 y)^{\sin\left(\frac{z+x}{2}\right)\cos\left(\frac{z-x}{2}\right)} + (\sin^2 z)^{\sin\left(\frac{x+y}{2}\right)\cos\left(\frac{x-y}{2}\right)} > 1$$

Daniel Sitaru

Solution (Tran Hong)

$$\sin\left(\frac{y+z}{2}\right)\cos\left(\frac{y-z}{2}\right) = \frac{1}{2}[\sin y + \sin z]$$

$$\sin\left(\frac{x+z}{2}\right)\cos\left(\frac{z-x}{2}\right) = \frac{1}{2}[\sin z + \sin x]$$

$$\sin\left(\frac{x+y}{2}\right)\cos\left(\frac{x-y}{2}\right) = \frac{1}{2}[\sin x + \sin y]$$

$$\Rightarrow LHS \geq (\sin x)^{(\sin y + \sin z)} + (\sin y)^{[\sin z + \sin x]} + (\sin z)^{[\sin x + \sin y]}$$

$$= (\sin x)^{\sin y}(\sin x)^{\sin z} + (\sin y)^{\sin z} \cdot (\sin y)^{\sin x} + (\sin z)^{\sin x} \cdot (\sin z)^{\sin y}$$

Let $X = \sin x\,; Y = \sin y\,; Z = \sin z$

$$\left(X, Y, Z \in \left(0; \frac{1}{2}\right)\right)$$

We prove that: $X^Y \cdot X^Z + Y^Z \cdot Y^X + Z^X \cdot Z^Y > 1$

Using AM-GM:

$$X^Y \cdot X^Z + Y^Z \cdot Y^X + Z^X \cdot Z^Y \geq 3\sqrt[3]{(X^Y \cdot Y^Z \cdot Z^X)(Y^X \cdot X^Z \cdot Z^X)}$$

But $X^Y\cdot Y^Z\cdot Z^X>\frac{1}{3\sqrt{3}}; Y^X\cdot X^Z\cdot Z^X>\frac{1}{3\sqrt{3}}$

Now, we want to prove: $X^Y\cdot Y^Z\cdot Z^X>\frac{1}{3\sqrt{3}}$ *(Similarly:* $Y^X\cdot X^Z\cdot Z^X>\frac{1}{3\sqrt{3}}$*)*

$$\Leftrightarrow Y\ln X+Z\ln Y+X\ln Z>-\ln(3\sqrt{3})$$

Using Jensen's inequality with $f(t)=\ln(\sin t), t\in\left(0,\frac{\pi}{2}\right)$

$$Y\ln X+Z\ln Y+X\ln Z\ge(X+Y+Z)\ln\left(\frac{XY+YZ+ZX}{X+Y+Z}\right)$$

$$>\left(\frac{1}{2}+\frac{1}{2}+\frac{1}{2}\right)\ln\left(\frac{\frac{1}{2}\cdot\frac{1}{2}+\frac{1}{2}\cdot\frac{1}{2}+\frac{1}{2}\cdot\frac{1}{2}}{\frac{1}{2}+\frac{1}{2}+\frac{1}{2}}\right)=\frac{3}{2}\ln\frac{1}{2}=-\frac{3}{2}\ln 2>-\ln(3\sqrt{3})$$

(Because $g(n)=n\ln\frac{\alpha}{n}\searrow\left(0;\frac{1}{2}\right)$*)*

Hence, $X^Y\cdot X^Z+Y^Z\cdot Y^X+Z^X+Z^X\cdot Z^Y>1$ *(Proved)*

6.20 If $x\in\mathbb{R}$ then:

$$\sin^{-1}(\sin^2 x)+\sin^{-1}(\cos^2 x)<\frac{49-9\cos 4x}{24}$$

Rovsen Pirguliyev

Solution (Tran Hong)

$$\sin^{-1}(\cos^2 x)+\sin^{-1}(\sin^2 x)\le\frac{49-9[8\cos^4 x-8\cos^2 x+1]}{24}\quad (*)$$

Let $u=\sin^{-1}(\cos^2 x), v=\sin^{-1}(\sin^2 x)\left(0\le u,v\le\frac{\pi}{2}\right)$

$$\Rightarrow\cos^2 x=\sin u;\sin^2 x=\sin v\Rightarrow\sin u+\sin v=1\Rightarrow 0<u+v\le\frac{\pi}{2}$$

(*) $\Leftrightarrow 24(u+v)\le 40-72\sin^2 u+72\sin u$

$$\Leftrightarrow 9\sin^2 u-9\sin u+3(u+v)-5\le 0\quad (**)$$

(**) true because: $-9\sin u\sin v+3(u+v)-5\le 0+3\cdot\frac{\pi}{2}-5<0$

6.21 If $x,y,z\in\mathbb{R}$ then:

$$\sqrt{3}(x^2+y^2+z^2)\ge|x^2+y^2-z^2|+2|z|(|x|+|y|)$$

Daniel Sitaru

Solution

$$(x^2+y^2+z^2)^2=(x^2+y^2-z^2)^2+(2yz)^2+(2xz)^2=$$

$$=\frac{(|x^2+y^2-z^2|)^2}{1}+\frac{(2|yz|)^2}{1}+\frac{(2|xz|)^2}{1}\ge$$

$$\overset{CBS}{\ge}\frac{\left(|x^2+y^2-z^2|+2|z|(|x|+|y|)\right)^2}{1+1+1}\overset{AM-GM}{\ge}\frac{(|x^2+y^2-z^2|+4|xyz|)^2}{3}$$

$$3(x^2+y^2+z^2)^2\ge(|x^2+y^2-z^2|+4|xyz|)^2$$

$$\sqrt{3}(x^2+y^2+z^2)\ge|x^2+y^2-z^2|+4|xyz|$$

6.22 If $a,b,c>0, a+b+c=1$ then:

$$\left(\frac{a+b}{2}\right)^{2ab}\cdot\left(\frac{b+c}{2}\right)^{2bc}\cdot\left(\frac{c+a}{2}\right)^{2ca}\ge a^{a-a^2}\cdot b^{b-b^2}\cdot c^{c-c^2}$$

Daniel Sitaru

Solution

$$\frac{a+b}{2}\overset{AM-GM}{\ge}\sqrt{ab}\Rightarrow\left(\frac{a+b}{2}\right)^{2ab}\ge\left(\sqrt{ab}\right)^{2ab}$$

$$\prod_{cyc}\left(\frac{a+b}{2}\right)^{2ab}\ge\prod_{cyc}\left(\sqrt{ab}\right)^{2ab}=\prod_{cyc}(ab)^{ab}=$$

$$=(ab)^{ab}\cdot(bc)^{bc}\cdot(ca)^{ca}=a^{ab}\cdot b^{ab}\cdot b^{bc}\cdot c^{bc}\cdot c^{ca}\cdot a^{ca}=$$

$$=a^{ab+ca}\cdot b^{ab+bc}\cdot c^{bc+ca}=a^{(b+c)a}\cdot b^{(a+c)b}\cdot c^{(b+a)c}=$$

$$=a^{(1-a)a}\cdot b^{(1-b)b}\cdot c^{(1-c)c}=a^{a-a^2}\cdot b^{b-b^2}\cdot c^{c-c^2}$$

6.23 If $A\in M_3(\mathbb{R}), Tr\,A=\det A=1$. Prove that:

$$\det(A^2+A+I_3)\ge 3Tr\,(A^{-1})$$

Marian Ursărescu

Solution (Șerban George Florin)

$$P(\lambda)=\lambda^3-(Tr\,A)\lambda^2+(Tr\,A^*)\lambda-\det A=-\det(A-\lambda I_3)$$

$$\varepsilon^3=1,\varepsilon^2+\varepsilon+1=0,\varepsilon=-\frac{1+i\sqrt{3}}{3}$$

$$\det(A^2+A+I_3)=\det(A-\varepsilon I_3)\cdot\det(A-\varepsilon^2 I_3)=P(\varepsilon)\cdot P(\varepsilon^2)=$$
$$=(1-(Tr\ A)\varepsilon^2+(Tr\ A^*)\varepsilon-\det A)(1-(Tr\ A)\varepsilon+(Tr\ A^*)\varepsilon^2-\det A)=$$
$$=(1-\varepsilon^2+(Tr\ A^*)\varepsilon-1)\cdot(1-\varepsilon+(Tr\ A^*)\varepsilon^2-1)=$$
$$=[-\varepsilon^2+(Tr\ A^*)\varepsilon]\cdot[-\varepsilon+(Tr\ A^*)\varepsilon^2]=\varepsilon^3-(Tr\ A^*)\varepsilon^4-$$
$$-(Tr\ A^*)\varepsilon^2+(Tr\ A^*)(Tr\ A^*)\varepsilon^3=1-\varepsilon\cdot(Tr\ A^*)-(Tr\ A^*)\varepsilon^2+$$
$$+(Tr\ A^*)^2=1-(Tr\ A^*)(\varepsilon+\varepsilon^2)+(Tr\ A^*)^2=(Tr\ A^*)^2+Tr\ A^*+1$$
$$A^*=(\det A)\cdot A^{-1}\Rightarrow A^*=A^{-1}\Rightarrow Tr\ A^*=Tr\ A^{-1}$$
$$\Rightarrow\det(A^2+A+I_3)=\left(Tr\ (A^{-1})\right)^2+Tr\ (A^{-1})+1\geq 3Tr\ (A^{-1})$$
$$\Leftrightarrow\left(Tr(A^{-1})\right)^2-2Tr(A^{-1})+1\geq 0\Leftrightarrow(Tr\ (A^{-1})-1)^2\geq 0$$

6.24 If $a,b,c\in(0,\pi)$ then:

$$\cos^2 a\cdot\cos^2 b\cdot\cos^2 c+(\sin a+\sin b+\sin c)^2>1$$

Daniel Sitaru

Solution (Tran Hong)

Let $x=\sin a\,;y=\sin b\,;z=\sin c.$

Because: $0<a,b,c<\pi\Rightarrow 0<x,y,z<1$

$$\prod\cos^2 a+\left(\sum\sin a\right)^2>1\Leftrightarrow\prod(1-\sin^2 a)+\left(\sum\sin a\right)^2>1$$
$$\Leftrightarrow(1-x^2)(1-y^2)(1-z^2)+(x+y+z)^2>1$$
$$\Leftrightarrow 1-(x^2+y^2+z^2)+(x^2y^2+y^2z^2+z^2x^2)-(xyz)^2+x^2+y^2+z^2+$$
$$+2(xy+xz+yz)>1$$

$$\Leftrightarrow(x^2y^2+y^2z^2+z^2x^2)+2(xy+yz+zx)-(xyz)^2>0$$

It is true because:

$$(x^2y^2+y^2z^2+z^2x^2)+2xy+2yz+2zx)\geq 6\sqrt[6]{2^3(xyz)^6}=6\sqrt{2}xyz$$

and: $6\sqrt{2}xyz>xyz>(xyz)^2\ \ (\because\ 0<xyz<1)$

6.25 If $0 < a, b, c \leq 16$ then:

$$27\exp\left(\sum_{cyc}\left(\sqrt{\frac{a+2b}{3}}-\sqrt{a}\right)\right) \leq \frac{(a+2b)(b+2c)(c+2a)}{abc}$$

Daniel Sitaru

Solution (Michael Sterghiou)

$$27\exp\left[\sum_{cyc}\left(\sqrt{\frac{a+2b}{3}}-\sqrt{a}\right)\right] \leq \frac{\prod_{cyc}(a+2b)}{abc} \quad \textit{(1)}$$

$$\textit{(1)} \to \sum_{cyc}\left(\sqrt{\frac{a+2b}{3}}-\sqrt{a}\right) \leq \left(\sum_{cyc}\ln\left(\frac{2a+b}{3}\right)\right) - \sum_{cyc}\ln a \textit{ or}$$

$$\sum_{cyc}(\sqrt{a}-\ln a) \geq \sum_{cyc}\left(\sqrt{\frac{2a+b}{3}}-\ln\left(\frac{2a+b}{3}\right)\right) \quad \textit{(2)}$$

The function $f(t) = \sqrt{t} - \ln t$ *on* $(0,16]$ *is convex as*

$f''(t) = \frac{4-\sqrt{t}}{4t^2} \geq 0$ *for* $0 < t \leq 16$. *Assume WLOG that* $a \geq b \geq c$.

Case I: $b \geq \frac{c+a}{2}$. *The triad* (a, b, c) *majorizes the triad* $\left(\frac{a+2b}{3} \geq \frac{c+2d}{3} \geq \frac{b+2c}{3}\right)$

as: $a \geq \frac{a+2b}{3}$

$a + b \geq \frac{a+2b}{3} + \frac{c+2a}{3} \leftrightarrow b \geq c$ *and* $a + b + c = \sum_{cyc}\frac{a+2b}{3}$

Case II: $b \leq \frac{c+a}{2}$. *The triad* $(a \geq b \geq c)$

majorizes the triad $\left(\frac{a+2a}{3} \geq \frac{a+2b}{3} \geq \frac{b+2c}{3}\right)$ *as*

$a \geq \frac{c+2a}{3}$ *and* $a + b \geq \frac{c+2a}{3} + \frac{a+2b}{3} \leftrightarrow b \geq c$. *Applying Karamata's inequality for the convex function* $f(t) = \sqrt{t} - \ln t$ *on* $(0,16]$ *for the above triads for either case I or II we obtain (2).*

6.26 If $-\frac{\pi}{2} \leq x, y, z \leq \frac{\pi}{2}$ then, prove that:

$$\sin^2 x \cdot \sin^2 y \cdot \sin^2 z + (\cos x + \cos y + \cos z)^2 \geq 1$$

Sudhir Jha-Kolkata (Inspired by Prof. Daniel Sitaru)

Solution(Tran Hong)

$$\textit{Putting: } u=\cos x\,;v=\cos y\,;w=\cos z$$

$$\left(x;y;z\in\left[-\frac{\pi}{2};\frac{\pi}{2}\right]\Rightarrow 0\le u,v,w\le 1\Rightarrow 0\le uvw\le 1\right)$$

$$\sin^2 x\sin^2 y\sin^2 z+(\cos x+\cos y+\cos z)^2\ge 1$$

$$\Leftrightarrow (1-\cos^2 x)(1-\cos^2 y)(1-\cos^2 z)+(\cos x+\cos y+\cos z)^2\ge 1$$

$$\Leftrightarrow (1-u^2)(1-v^2)(1-z^2)+(u+v+w)^2\ge 1$$

$$\Leftrightarrow u^2v^2+v^2w^2+w^2u^2+2(uv+vw+wu)\ge (uvw)^2$$

$$\because u^2v^2+v^2w^2+w^2u^2+2(uv+vw+wu)\overset{AM-GM}{\ge} 6\sqrt[6]{2(uvw)^6}=6\sqrt[6]{2}uvw$$

$$\because 6\sqrt[6]{2}uvw\ge uvw\ge (uvw)^2 \textit{ (Because: } 0\le uvw\le 1\textit{) Proved.}$$

6.27 If $0<d<c<b<a<\frac{\pi}{2}$ then:

$$\csc\left(\frac{\pi b}{2a}\right)\cdot\csc\left(\frac{\pi c}{2b}\right)\cdot\csc\left(\frac{\pi d}{2c}\right)<\frac{\sin a}{\sin d}$$

Daniel Sitaru

Solution(Tran Hong)

$$\csc\left(\frac{\pi a}{2b}\right)=\frac{1}{\sin\left(\frac{\pi a}{2b}\right)}\le\frac{1}{\frac{2}{\pi}\left(\frac{\pi a}{2b}\right)}=\frac{b}{a}$$

$$\csc\left(\frac{\pi b}{2c}\right)=\frac{1}{\sin\left(\frac{\pi b}{2c}\right)}\le\frac{1}{\frac{2}{\pi}\cdot\left(\frac{\pi}{2c}\right)}=\frac{c}{b}$$

$$\csc\left(\frac{\pi c}{2d}\right)=\frac{1}{\sin\left(\frac{\pi c}{2d}\right)}\le\frac{1}{\frac{2}{\pi}\cdot\left(\frac{\pi c}{2d}\right)}=\frac{d}{c}$$

$$\to\csc\left(\frac{\pi b}{2a}\right)\cdot\csc\left(\frac{\pi c}{2b}\right)\cdot\csc\left(\frac{\pi d}{2c}\right)\le\frac{b}{a}\cdot\frac{c}{b}\cdot\frac{d}{c}=\frac{d}{a}$$

$$\textit{Now, we must show that: }\frac{d}{a}<\frac{\sin a}{\sin d}\leftrightarrow d\sin d<a\sin a\quad (*)$$

$$\textit{Let } f(x)=x\sin x\quad\left(0<x<\frac{\pi}{2}\right)\to f'(x)=\sin x+x\cos x>0\quad\left(0<x<\frac{\pi}{2}\right)$$

$$\textit{Hence, } f(x)\nearrow\left(0;\frac{\pi}{2}\right)$$

Because $0 < d < a < \frac{\pi}{2} \to f(d) < f(a) \to d\sin d < a\sin a \to$ *(*) true.*

6.28 Prove that:

$$a^{\cos^2 x} + (1-a)\cos^2 x \le 1, a \ge 0$$

Mohammed Bouras

Solution (Tran Hong)

$$Let: f(\alpha) = log\alpha - \alpha + 1\ (\alpha > 0), \qquad f'(\alpha) = \frac{1}{\alpha} - 1 = \frac{1-\alpha}{\alpha}$$

$$f'(\alpha) = 0 \Leftrightarrow \alpha = 1, f(\alpha) \le f(1) = 0 \Rightarrow log\alpha \le \alpha - 1 \dots (1)$$

$$\text{Let: } f(\beta) = \beta log\beta + 1 - \beta\ ; (\beta > 1), f'(\beta) = log\beta > 0 \Rightarrow \frac{\beta-1}{log\beta} < \beta \dots (2)$$

$$\text{Let: } t = cos^2 t\ ; (0 < t \le 1). \text{ Put: } \varphi(t) = a^t + (1-a)t; (0 < t \le 1)$$

$$\therefore a = 0 \Rightarrow \varphi(t) = t \le 1; (0 < t \le 1)$$

$$\therefore 0 < a < 1 \Rightarrow \varphi'(t) = a^t loga + (1-a);\ \varphi'(t) = 0 \Leftrightarrow a^t = \frac{a-1}{loga} \Leftrightarrow$$

$$t = \log_a\left(\frac{a-1}{loga}\right) \underset{0<a<1}{\overset{(1)}{<}} 0 \Rightarrow t \notin (0;1]$$

$$\Rightarrow \varphi'(t) < 0 \Rightarrow \varphi(t) \searrow (0;1] \Rightarrow \varphi(t) < \varphi(0) < 1$$

$$\therefore a = 1 \Rightarrow \varphi(t) = 1 \le 1\ \ (true),\ \therefore a > 1 \Rightarrow \varphi'(t) = a^t loga + (1-a);$$

$$\varphi'(t) = 0 \Leftrightarrow t = \gamma = \log_a\left(\frac{a-1}{loga}\right) \underset{a>1}{\overset{(1)}{>}} 0;\ and\ t = \gamma < 1 \overset{(2)}{\Rightarrow}$$

$$t \in (0,1) \Rightarrow \varphi(t) < \varphi(1) = 1. \text{ So, } a^t + (1-a)t \le 1;\ \forall a \ge 0)$$

6.29 If $a, b, c \ge 1$ then:

$$\frac{\left(a^{\frac{1}{a}} + b^{\frac{1}{b}} - c^{\frac{1}{c}}\right)\left(b^{\frac{1}{b}} + c^{\frac{1}{c}} - a^{\frac{1}{a}}\right)\left(c^{\frac{1}{c}} + a^{\frac{1}{a}} - b^{\frac{1}{b}}\right)}{a^{\frac{1}{a}} \cdot b^{\frac{1}{b}} \cdot c^{\frac{1}{c}}} \le 8$$

Daniel Sitaru

Solution

$$[a] \le a < [a] + 1; [*] - \textit{great integer function}$$

$$2^a \ge 2^{[a]} = (1+1)^{[a]} \overset{BERNOULLI}{>} 1 + [a] > a$$

$$2^a > a \Rightarrow (2^a)^{\frac{1}{a}} > a^{\frac{1}{a}} \ge 1 \Rightarrow 2 > a^{\frac{1}{a}} \ge 1$$

$$\textit{Analogous: } 2 > b^{\frac{1}{b}} \ge 1; 2 > c^{\frac{1}{c}} \ge 1$$

$$a^{\frac{1}{a}} + b^{\frac{1}{b}} \ge 1 + 1 = 2 > c^{\frac{1}{c}}$$

$$a^{\frac{1}{a}} + b^{\frac{1}{b}} > c^{\frac{1}{c}}$$

$$\textit{Analogous: } b^{\frac{1}{b}} + c^{\frac{1}{c}} > a^{\frac{1}{a}}; c^{\frac{1}{c}} + a^{\frac{1}{a}} > b^{\frac{1}{b}}$$

$a^{\frac{1}{a}}; b^{\frac{1}{b}}; c^{\frac{1}{c}}$ *can be sides in a triangle.*

In a triangle with sides x, y, z *by Padoa's inequality:*

$$(x+y-z)(y+z-x)(z+x-y) \le 8xyz$$

$$\frac{(x+y-z)(y+z-x)(z+x-y)}{xyz} \le 8$$

$$\textit{For } x = a^{\frac{1}{a}}; y = b^{\frac{1}{b}}; z = c^{\frac{1}{c}}$$

$$\frac{\left(a^{\frac{1}{a}} + b^{\frac{1}{b}} - c^{\frac{1}{c}}\right)\left(b^{\frac{1}{b}} + c^{\frac{1}{c}} - a^{\frac{1}{a}}\right)\left(c^{\frac{1}{c}} + a^{\frac{1}{a}} - b^{\frac{1}{b}}\right)}{a^{\frac{1}{a}} \cdot b^{\frac{1}{b}} \cdot c^{\frac{1}{c}}} \le 8$$

Equality holds for $a = b = c = 1$.

6.30 If $a, b, c, d > 0; a^2cd + b^2da + c^2ab + d^2bc = 4abcd$ then:

$$\frac{a^2}{b^2}\left(\frac{a}{b} - 1\right) + \frac{b^2}{c^2}\left(\frac{b}{c} - 1\right) + \frac{c^2}{d^2}\left(\frac{c}{d} - 1\right) + \frac{d^2}{a^2}\left(\frac{d}{a} - 1\right) \ge 0$$

Daniel Sitaru

Solution

$$\textit{If } u > 0 \Rightarrow (u-1)^2(u+1) \ge 0$$

$$(u^2 - 2u + 1)(u + 1) \ge 0$$

$$u^3+u^2-2u^2-2u+u+1\geq 0$$

$$u^3-u^2-u+1\geq 0\Rightarrow u^3\geq u^2+u-1$$

For $u=\frac{a}{b}$ *and successively* $u=\frac{b}{c};u=\frac{c}{d};u=\frac{a}{b}$ *then:*

$$\left(\frac{a}{b}\right)^3\geq\left(\frac{a}{b}\right)^2+\left(\frac{a}{b}\right)-1\quad (1)$$

$$\left(\frac{b}{c}\right)^3\geq\left(\frac{b}{c}\right)^2+\left(\frac{b}{c}\right)-1\quad (2)$$

$$\left(\frac{c}{d}\right)^3\geq\left(\frac{c}{d}\right)^2+\left(\frac{c}{d}\right)-1\quad (3)$$

$$\left(\frac{d}{a}\right)^3\geq\left(\frac{d}{a}\right)^2+\left(\frac{d}{a}\right)-1\quad (4)$$

By adding (1); (2); (3); (4):

$$\sum_{cyc}\left(\frac{a}{b}\right)^3\geq\sum_{cyc}\left(\frac{c}{d}\right)^2+\sum_{cyc}\frac{a}{b}-4=\sum_{cyc}\left(\frac{c}{d}\right)^2+\frac{a}{b}+\frac{b}{c}+\frac{c}{d}+\frac{d}{a}-4=$$

$$=\sum_{cyc}\left(\frac{a}{b}\right)^2+\frac{a^2cd+b^2da+c^2ab+d^2bc-4abcd}{abcd}=$$

$$=\sum_{cyc}\left(\frac{a}{b}\right)^2+\frac{4abcd-4abcd}{abcd}=\sum_{cyc}\left(\frac{a}{b}\right)^2$$

6.31 If $m,n,p,q\in\mathbb{N};m,n,p,q\geq 4$ then:

$$\mathbf{4^n(4^n+1)+4^m(4^m+1)+4^p(4^p+1)+4^q(4^q+1)}$$
$$\mathbf{\geq 4mnpq(mnpq+1)}$$

Daniel Sitaru

Solution

We prove by induction: $4^n\geq n^4;n\geq 4$

For $n=4\Rightarrow 4^4\geq 4^4$ *(True).* $P(n):4^n\geq n^4$ *(True)*

$P(n+1):4^{n+1}\geq(n+1)^4$ *(to prove)*

$4^{n+1}=4^n\cdot 4\geq 4n^4\geq(n+1)^4$ *(to prove)*

$$\sqrt[4]{n^4\cdot 4}\geq\sqrt[4]{(n+1)^4}\Leftrightarrow n\sqrt{2}\geq n+1\Leftrightarrow n\left(\sqrt{2}-1\right)\geq 1$$

$$But\ n \geq 4 \Rightarrow n(\sqrt{2}-1) \geq 4(\sqrt{2}-1) \geq 1 \Leftrightarrow$$

$$4\sqrt{2} \geq 5 \Leftrightarrow 32 > 25; P(n) \to P(n+1)$$

$m,n,p,q \geq 4 \Rightarrow 4^n \geq n^4; 4^m \geq m^4; 4^p \geq p^4; 4^q \geq q^4$. *By adding:*

$$4^n + 4^m + 4^p + 4^q \geq n^4 + m^4 + p^4 + q^4 \overset{AM-GM}{\geq}$$

$$\geq 4\sqrt[4]{n^4 \cdot m^4 \cdot p^4 \cdot q^4} = 4mnpq \quad (1)$$

$$\sqrt{\frac{16^n + 16^m + 16^p + 16^q}{4}} \overset{QM-AM}{\geq} \frac{4^n + 4^m + 4^p + 4^q}{4} \overset{(1)}{\geq} mnpq$$

By squaring: $16^n + 16^m + 16^p + 16^q \geq 4m^2n^2p^2q^2 \quad (2)$

By adding (1); (2):

$$16^n + 4^n + 16^m + 4^m + 16^p + 4^p + 16^q + 4^q \geq 4m^2n^2p^2q^2 + 4mnpq$$

$$4^n(4^n+1) + 4^m(4^m+1) + 4^p(4^p+1) + 4^q(4^q+1)$$

$$\geq 4mnpq(mnpq+1)$$

6.32 If $0 \leq x \leq \frac{\pi}{2}, 0 \leq y \leq 1$ then:

$$2x \leq \pi xy\cos\frac{x}{2} + \pi(1-y)\sin x \leq \pi x$$

Daniel Sitaru

Solution (Tran Hong)

For $x \in \left[0, \frac{\pi}{2}\right]$ *we have:* $\frac{2}{\pi}x \leq \sin x \leq x,\ \frac{2}{\pi} \overset{(1)}{\leq} \cos\frac{x}{2} \leq 1$

Let: $\varphi(x) = \cos\frac{x}{2} - \frac{2}{\pi}, \forall x \in \left[0, \frac{\pi}{2}\right]$

$$\varphi'(x) = -\frac{1}{2}\sin\frac{x}{2} < 0, \forall \in \left[0, \frac{\pi}{2}\right] \text{ then } \varphi(x) \searrow \text{ on } \left[0, \frac{\pi}{2}\right]$$

$$\varphi(x) \geq \varphi\left(\frac{\pi}{2}\right) = \cos\frac{\pi}{4} - \frac{2}{\pi} = \frac{\sqrt{2}}{2} - \frac{2}{\pi} > 0 \Rightarrow (1)true.$$

Now,

$$\pi xy + \pi(1-y)\sin x \overset{0 \leq y \leq 1}{\geq} \pi xy \cdot \frac{2}{\pi} + \pi(1-y) \cdot \frac{2}{\pi} \cdot x = 2x$$

$$\pi xy\cos\frac{x}{2}+\pi(1-y)\sin x\leq \pi xy+\pi(1-y)x=\pi x$$

6.33 If $x\geq 0$ then:

$$(x^2+x+1)^{x^2+2}+e^x\geq 2+3x+3x^2+3x^3$$

Nguyen Van Canh

Solution (Sanong Huayrerai)

For $x\geq 0$ *we have:*

$$(x^2+x+1)^{x^2+2}+e^x=(x^2+x+1)^{x^2+1}(x^2+x+1)+e^x$$
$$\geq\left((x^2+1)(x^2+x)+1\right)(x^2+x+1)+e^x$$
$$=(x^4+x^3+x^2+x+1)(x^2+x+1)+e^x$$
$$=x^6+2x^5+3x^4+3x^3+3x^2+2x+1+e^x$$
$$\geq x^6+2x^5+3x^4+3x^3+3x^2+2x+1+x+1$$
$$\geq x^6+2x^5+3x^4+3x^3+3x^2+3x+2\geq 2+3x+3x^2+3x^3$$

6.34 If $x,y>0$, then:

$$\sqrt{\frac{x^2+y^2}{8}}+\sqrt{xy}\leq\frac{3}{4}(x+y)$$

George Apostolopoulos

Solution(Daniel Sitaru)

$$\text{Let: } u=\sqrt{\tfrac{x^2+y^2}{2}};\ v=\sqrt{xy}\Rightarrow 2u^2=x^2+y^2;\ v^2=xy$$
$$x^2+y^2+2xy=2u^2+2v^2\Rightarrow(x+y)^2=2(u^2+v^2)$$

$$0\leq(u-v)^2\Rightarrow(u+v)^2\leq 2(u^2+v^2)\Rightarrow(u+v)^2\leq(x+y)^2$$
$$u+v\leq x+y\Rightarrow\sqrt{\frac{x^2+y^2}{2}}+\sqrt{xy}\leq x+y\quad(1)$$
$$\sqrt{xy}\leq\frac{x+y}{2}\quad(2)$$

By (1)+(2) we have: $\sqrt{\frac{x^2+y^2}{2}}+2\sqrt{xy}\le\frac{3}{2}(x+y)$

$$\frac{1}{2}\sqrt{\frac{x^2+y^2}{2}}+\sqrt{xy}\le\frac{3}{4}(x+y)\Rightarrow\sqrt{\frac{x^2+y^2}{8}}+\sqrt{xy}\le\frac{3}{4}(x+y)$$

6.35 If $a,b,c\in\mathbb{N}\setminus\{0,1,2,3\}$ then:

$$b^2\cdot\sqrt[a]{a}+c^2\cdot\sqrt[b]{b}+a^2\cdot\sqrt[c]{c}\ge 48\sqrt{2}$$

Daniel Sitaru

Solution

By induction we prove $4^n\ge n^4; n\ge 4$

For $n=4\Rightarrow 4^4\ge 4^4$ *(true)*

$P(n): 4^n\ge n^4$ *(true)*

$P(n+1): 4^{n+1}\ge(n+1)^4$ *(to prove)*

$4^{n+1}=4^n\cdot 4\ge 4n^4\ge(n+1)^4$ *(to prove)*

$$\sqrt[4]{4n^4}\ge\sqrt[4]{(n+1)^4}\Rightarrow n\sqrt{2}\ge n+1$$

$$n(\sqrt{2}-1)\ge 4(\sqrt{2}-1)>1\Leftrightarrow 4\sqrt{2}>5\Leftrightarrow 32>25$$

$$P(n)\to P(n+1)$$

If $a,b,c\ge 4\Rightarrow\sqrt[a]{a}\ge\sqrt[4]{4}=\sqrt{2};\ \sqrt[b]{b}\ge\sqrt{2};\ \sqrt[c]{c}\ge\sqrt{2}\Rightarrow$

$$\Rightarrow\frac{1}{\sqrt[a]{a}}\le\frac{1}{\sqrt{2}};\frac{1}{\sqrt[b]{b}}\le\frac{1}{\sqrt{2}};\frac{1}{\sqrt[c]{c}}\le\frac{1}{\sqrt{2}}$$

$\frac{1}{\sqrt[a]{a}}+\frac{1}{\sqrt[b]{b}}+\frac{1}{\sqrt[c]{c}}\le\frac{3}{\sqrt{2}}$ *(1)*

$$b^2\cdot\sqrt[a]{a}+c^2\cdot\sqrt[b]{b}+a^2\cdot\sqrt[c]{c}=\frac{b^2}{\frac{1}{\sqrt[a]{a}}}+\frac{c^2}{\frac{1}{\sqrt[b]{b}}}+\frac{a^2}{\frac{1}{\sqrt[c]{c}}}\overset{BERGSTROM}{\ge}$$

$$\ge\frac{(a+b+c)^2}{\frac{1}{\sqrt[a]{a}}+\frac{1}{\sqrt[b]{b}}+\frac{1}{\sqrt[c]{c}}}\overset{(1)}{\ge}\frac{(4+4+4)^2}{\frac{3}{\sqrt{2}}}=\frac{9\cdot 16\cdot\sqrt{2}}{3}$$

$$b^2\cdot\sqrt[a]{a}+c^2\cdot\sqrt[b]{b}+a^2\cdot\sqrt[c]{c}\ge 48\sqrt{2}$$

6.36 If $x,y,z>1;xyz=8$ then:

$$\left(\frac{x}{2}\right)^x+\left(\frac{y}{2}\right)^y+\left(\frac{z}{2}\right)^z\geq 3$$

Daniel Sitaru

Solution

$$\left(\frac{x}{2}\right)^x=\left(1+\left(\frac{x}{2}-1\right)\right)^x\overset{BERNOULLI}{\geq}1+x\left(\frac{x}{2}-1\right)\geq x-1$$

$$\Leftrightarrow 1+\frac{x^2}{2}-x\geq x-1$$

$$\frac{x^2}{2}-2x+2\geq 0\Leftrightarrow x^2-4x+4\geq 0\Leftrightarrow (x-2)^2\geq 0$$

$$\left(\frac{x}{2}\right)^x\geq x-1,\left(\frac{y}{2}\right)^y\geq y-1,\left(\frac{z}{2}\right)^z\geq z-1$$

By adding: $\left(\frac{x}{2}\right)^x+\left(\frac{y}{2}\right)^y+\left(\frac{z}{2}\right)^z\geq x+y+z-3\geq$

$$\overset{AM-GM}{\geq}3\sqrt[3]{xyz}-3=3\sqrt[3]{8}-3=3\cdot 2-3=3$$

Equality holds for $a=b=c=2$.

6.37 If $0<y<x<2y$ then:

$$x(x+y)>3(x-y)\sqrt{3(4y^2-x^2)}$$

Daniel Sitaru

Solution

Let's consider $\Delta ABC; BC=a=xy; AC=b=y^2; AB=c=x^2-y^2$.

$$a<b+c\Leftrightarrow xy<x^2-y^2+y^2\Leftrightarrow y<x\ (true)$$

$$b<a+c\Leftrightarrow y^2<xy+x^2-y^2\Leftrightarrow x(x+y)>2y^2$$

But: $x(x+y)>y(y+y)=2y^2$

$$c<a+b\Leftrightarrow x^2-y^2<xy+y^2\Leftrightarrow x^2<y(x+2y)$$

$$\Leftrightarrow 2y^2+xy-x^2>0\Leftrightarrow 2t^2-t-1>0; t=\frac{y}{x}$$

$$\Delta = 9; t_1 = \frac{1+3}{2} = 2; t_2 = \frac{1-3}{2} = -1, \min(2t^2 - t - 1) = -\frac{9}{8}$$

$$t > 1 \Rightarrow 2t^2 - t - 1 > 2 \cdot 1^2 - 1 - 1 = 0 \quad \textit{(true)}$$

Denote s – semiperimeter; F – area; r – inradii

$$s = \frac{a+b+c}{2} = \frac{xy+y^2+x^2-y^2}{2} = \frac{x^2+xy}{2} \quad (1)$$

$$\cos B = \frac{a^2+c^2-b^2}{2ac} = \frac{x^2y^2+(x^2-y^2)^2-y^4}{2xy(x^2-y^2)} =$$

$$= \frac{x^4-x^2y^2+y^4-y^4}{2xy(x^2-y^2)} = \frac{x^2(x^2-y^2)}{2xy(x^2-y^2)} = \frac{x}{2y} < 1 \quad \textit{(true)}$$

$$\sin B = \sqrt{1-\cos^2 B} = \sqrt{1-\frac{x^2}{4y^2}} = \frac{\sqrt{4y^2-x^2}}{2y} \quad (2)$$

$$F = \frac{1}{2}ac\sin B = \frac{1}{2}\cdot xy\cdot(x^2-y^2)\cdot\frac{\sqrt{4y^2-x^2}}{2y}$$

$$F = \frac{x(x^2-y^2)\sqrt{4y^2-x^2}}{4}$$ *. By Mitrinovic's inequality:*

$s > 3\sqrt{3}r$ *(Equality doesn't hold because ΔABC can't be an equilateral one*

$x < y \Rightarrow xy < y^2 \Rightarrow a < b$*);* $s > 3\sqrt{3}\cdot\frac{F}{s} \Rightarrow s^2 > 3\sqrt{3}F$

By (1); (2): $\frac{x^2(x+y)^2}{4} > 3\sqrt{3}\cdot\frac{x(x-y)(x+y)\sqrt{4y^2-x^2}}{4}$

$$x(x+y) > 3\sqrt{3}(x-y)\sqrt{4y^2-x^2}, \qquad \frac{x(x+y)}{3(x-y)} > \sqrt{3(4y^2-x^2)}$$

$$x(x+y) > 3(x-y)\sqrt{3(4y^2-x^2)}$$

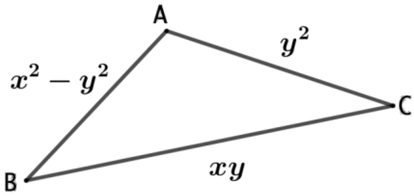
6.38 If $0 < y < x < 2y$ then:

$$\boldsymbol{x(x+y)\sqrt{4y^2-x^2} < 3y^3\sqrt{3}}$$

Daniel Sitaru

Solution

Let's consider $\Delta ABC; BC = a = xy; AC = b = y^2; AB = c = x^2 - y^2$



$$a < b + c \Leftrightarrow xy < x^2 - y^2 + y^2 \Leftrightarrow y < x \quad \textit{(true)}$$

$$b < a + c \Leftrightarrow y^2 < xy + x^2 - y^2 \Leftrightarrow x(x+y) > 2y^2$$

But: $x(x+y) > y(y+y) = 2y^2$

$$c < a + b \Leftrightarrow x^2 - y^2 < xy + y^2 \Leftrightarrow x^2 < y(x+2y)$$

$$\Leftrightarrow 2y^2 + xy - x^2 > 0 \Leftrightarrow 2t^2 - t - 1 > 0; t = \frac{y}{x}$$

$$\Delta = 9; t_1 = 2; t_2 = -1; \min(2t^2 - t - 1) = -\frac{9}{8}$$

$$t > 1 \Rightarrow 2t^2 - t - 1 > 2 \cdot 1^2 - 1 - 1 = 0 \quad \textit{(true)}$$

Denote s – semiperimeter; R – circumradii

$$s = \frac{a+b+c}{2} = \frac{xy+y^2+x^2-y^2}{2} = \frac{x^2+xy}{2} \quad (1)$$

$$\cos B = \frac{a^2 + c^2 - b^2}{2ac} = \frac{x^2y^2 + (x^2-y^2)^2 - y^4}{2xy(x^2-y^2)} =$$

$$= \frac{x^4 - x^2y^2 + y^4 - y^4}{2xy(x^2-y^2)} = \frac{x^2(x^2-y^2)}{2xy(x^2-y^2)} = \frac{x}{2y} < 1 \quad \textit{(true)}$$

$$\sin B = \sqrt{1 - \cos^2 B} = \sqrt{1 - \frac{x^2}{4y^2}} = \frac{\sqrt{4y^2 - x^2}}{2y}$$

$$R = \frac{b}{2\sin B} = \frac{y^2}{2 \cdot \frac{\sqrt{4y^2 - x^2}}{2y}} = \frac{y^3}{\sqrt{4y^2 - x^2}}$$

By Mitrinovic's inequality: $s \le \frac{3\sqrt{3}}{2}R$. *By (1); (2):* $\frac{x(x+y)}{2} < \frac{3\sqrt{3}}{2} \cdot \frac{y^3}{\sqrt{4y^2-x^2}}$

$$x(x+y)\sqrt{4y^2 - x^2} < 3y^3\sqrt{3}$$

(Equality doesn't hold because ΔABC can't be an equilateral one:

$$x < y \Rightarrow xy < y^2 \Rightarrow a < b)$$

6.39 If $2 < a \le b$ then:

$$\frac{\log(a+b)-\log 2}{\log(a+b-2)-\log 2} \le \frac{\log(ab)}{2\log(\sqrt{ab}-1)}$$

Daniel Sitaru

Solution (Sudhir Jha)

Let be the function: $f(x) = \frac{\log x}{\log(x-1)}, x > 2, f'(x) = \frac{(x-1)\log(x-1) - x\log x}{x(x-1)\log^2(x-1)} < 0$

f-is decreasing for all x>2, then:

$$f\left(\frac{a+b}{2}\right) \overset{\frac{a+b}{2} \ge \sqrt{ab}}{\lesssim} f(\sqrt{ab}) \Leftrightarrow \frac{\log\left(\frac{a+b}{2}\right)}{\log\left(\frac{a+b}{2}-1\right)} \le \frac{\log\sqrt{ab}}{\log(\sqrt{ab}-1)} \Leftrightarrow$$

$$\frac{\log(a+b)-\log 2}{\log(a+b-2)-\log 2} \le \frac{\log(ab)}{2\log(\sqrt{ab}-1)}$$

6.40 If $a, b, c, d, e, m, n \in \mathbb{R}, m, n \ge 0, k \in \mathbb{N}$ prove:

$$\sqrt[k]{m|a-b|+n|c-d|} + \sqrt[k]{m|b-e|+n|d-f|} \ge \sqrt[k]{m|a-e|+n|c-f|}$$

Jalil Hajimir

Solution (Tran Hong)

Let: $u = \sqrt[k]{m|a-b|+n|c-d|}; u \ge 0 \Rightarrow u^k = m|a-b|+n|c-d|$

Let: $v = \sqrt[k]{m|c-d|+n|e-f|}; v \ge 0 \Rightarrow v^k = m|c-d|+n|e-f|$

$$u^k + v^k = m|a-b| + (m+n)|c-d| + n|e-f| \Rightarrow$$

$$m|a-b| + n|e-f| = u^k + v^k - (m+n)|c-d|$$

We need to prove:

$$u + v \ge \sqrt[k]{u^k + v^k - (m+n)|c-d|} \Leftrightarrow$$

$$(u+v)^k \ge u^k + v^k - (m+n)|c-d|$$

$$u^k + v^k + \sum_{i=1}^{k-1} \binom{k}{i} u^i v^{k-i} \ge u^k + v^k - (m+n)|c-d| \Leftrightarrow$$

$$\sum_{i=1}^{k-1}\binom{k}{i}u^i v^{k-i} \geq -(m+n)|c-d|$$

Which is true because:

$$u, v, m, n, |c-d| \geq 0 \Rightarrow \sum_{i=1}^{k-1}\binom{k}{i}u^i v^{k-i} \geq -(m+n)|c-d|.$$

6.41 If $a, b, c \in (0,1)$ or $a, b, c \in (1,\infty)$, $ab+bc+ca=abc$ then:

$$2\cdot\sqrt[4]{\left(\prod_{cyc}\tan^{-1}a\right)\left(\sum_{cyc}\tan^{-1}a\right)} \leq \tan^{-1}\left(\frac{\sqrt{(\sum_{cyc}a^2)(\sum_{cyc}(1-a)^2)}}{1-abc}\right)$$

Florică Anastase

Solution (Adrian Popa)

$$2\sqrt[4]{\tan^{-1}a\cdot\tan^{-1}b\cdot\tan^{-1}c(\tan^{-1}a+\tan^{-1}b+\tan^{-1}c)}$$

$$\overset{Am-Gm}{\leq} 2\cdot\frac{2(\tan^{-1}a+\tan^{-1}b+\tan^{-1}c)}{4}$$

$$=\tan^{-1}\left(\frac{a+b+c-abc}{1-ab-bc-ca}\right)=\tan^{-1}\left(\frac{a+b+c-ab-bc-ca}{1-abc}\right)$$

$$=\tan^{-1}\left(\frac{a(1-b)+b(1-c)+c(1-a)}{1-abc}\right)$$

Let be the function: $f(x)=\tan^{-1}x, f'(x)=\frac{1}{1+x^2}>0\forall x\in\mathbb{R}\Rightarrow$

f *–increasing*

$$a(1-b)+b(1-c)+c(1-a)\overset{C.B.S}{\leq}$$

$$\leq\sqrt{(a^2+b^2+c^2)((1-a)^2+(1-b)^2+(1-c)^2)}$$

$$If\ a,b,c\in(0,1)\Rightarrow\begin{cases}1-a>0\\1-b>0\\1-c>0\end{cases} and\ 1-abc>0\Rightarrow\frac{\sum a(1-b)}{1-abc}>0$$

$$If\ a,b,c\in(1,\infty))\Rightarrow\begin{cases}1-a<0\\1-b<0\\1-c<0\end{cases} and\ 1-abc<0\Rightarrow\frac{\sum a(1-b)}{1-abc}>0$$

6.42 If $a > 2, 0 \le x \le y \le z, x + y + z = 3$ then:

$$(x-1)\log a\log(a-1) + (y-1)\log(a-1)\log(a+1) + (z-1)\log a\log(a+1) \ge 0$$

Daniel Sitaru

Solution (Tran Hong)

For $x > 0$ *we have:* $\varphi(x) = \log x$ *increasing on* $(1, \infty)$

$$\overset{a>2}{\Longrightarrow} 0 < \log(a-1)\log a \le \log(a-1)\log(a+1) \le \log a\log(a+1) \quad (1)$$

$$0 \le x \le y \le z; x + y + z = 3 \Rightarrow z \ge 1$$

$$\textit{Case1: } x \ge 1 \overset{y\ge x}{\Longrightarrow} y \ge 1$$

$$\Rightarrow (x-1)\log a\log(a-1) + (y-1)\log(a-1)\log(a+1) + (z-1)\log a\log(a+1) \ge 0$$

$$\textit{Case 2: } 0 \le x \le 1 \overset{y\ge x}{\Longrightarrow} \begin{cases} y \ge 1 \ge x \\ 0 \le x \le y \le 1 \end{cases}$$

If $0 \le x \le y \le 1$ *then:*

$$(x-1)\log a\log(a-1) + (y-1)\log(a-1)\log(a+1) + (z-1)\log a\log(a+1)$$
$$\ge (x-1)\log a\log(a+1) + (y-1)\log a\log(a+1) + (z-1)\log a\log(a+1) = (x+y+z-3)\log a\log(a+1) = 0$$

If $0 \le x \le 1 \le y$ then

$$(x-1)\log a\log(a-1) + (y-1)\log(a-1)\log(a+1) + (z-1)\log(a-1)\log(a+1)$$
$$\ge (x-1)\log(a-1)\log(a+1) + (y-1)\log(a-1)\log(a+1) + (z-1)\log(a-1)\log(a+1) = (x+y+z-3)\log(a-1)\log(a+1) = 0$$

6.43 If $a, b, c > 0; a + b + c = 3$ then:

$$(a+c)e^{\frac{1}{a}} + b(b+a)e^{\frac{1}{b}} + c(c+b)e^{\frac{1}{c}} \ge 9e$$

Daniel Sitaru

Solution

Let be $f:(0,\infty)\to\mathbb{R}; f(x)=e^x; f'(x)=e^x; f''(x)=e^x>0; f$ *convexe. By Jensen's inequality:*

$$e^{\lambda_1 x_1+\lambda_2 x_2+\lambda_3 x_3}\le \lambda_1 e^{x_1}+\lambda_2 e^{x_2}+\lambda_3 e^{x_3} \quad (1)$$

$$\lambda_1+\lambda_2+\lambda_3=1; \lambda_1,\lambda_2,\lambda_3>0$$

We take:

$$\lambda_1=\frac{a^2+2ac}{(a+b+c)^2}; \lambda_2=\frac{b^2+2ba}{(a+b+c)^2}; \lambda_3=\frac{c^2+2cb}{(a+b+c)^2}$$

$$\lambda_1+\lambda_2+\lambda_3=\frac{a^2+2ac+b^2+2ba+c^2+2cb}{(a+b+c)^2}=\frac{(a+b+c)^2}{(a+b+c)^2}=1$$

$$x_1=\frac{1}{a}; x_2=\frac{1}{b}; x_3=\frac{1}{c}$$

Replace in (1):

$$e^{\frac{a^2+2ac}{(a+b+c)^2}\cdot\frac{1}{a}+\frac{b^2+2ba}{(a+b+c)^2}\cdot\frac{1}{b}+\frac{c^2+2cb}{(a+b+c)^2}\cdot\frac{1}{c}}\le$$

$$\le\frac{a^2+2ac}{(a+b+c)^2}e^{\frac{1}{a}}+\frac{b^2+2ba}{(a+b+c)^2}e^{\frac{1}{b}}+\frac{c^2+2cb}{(a+b+c)^2}e^{\frac{1}{c}}$$

$$e^{\frac{a+2c+b+2a+c+2b}{(a+b+c)^2}}\le\frac{a^2+2ac}{(a+b+c)^2}e^{\frac{1}{a}}+\frac{b^2+2ba}{(a+b+c)^2}e^{\frac{1}{b}}+\frac{c^2+2cb}{(a+b+c)^2}e^{\frac{1}{c}}$$

$$e^{\frac{3}{a+b+c}}\cdot(a+b+c)^2\le(a^2+2ac)e^{\frac{1}{a}}+(b^2+2ba)e^{\frac{1}{b}}+(c^2+2cb)e^{\frac{1}{c}}$$

$$e^{\frac{3}{3}}\cdot 3^2\le a(a+c)e^{\frac{1}{a}}+b(b+a)e^{\frac{1}{b}}+c(c+b)e^{\frac{1}{c}}$$

$$a(a+c)e^{\frac{1}{a}}+b(b+a)e^{\frac{1}{b}}+c(c+b)e^{\frac{1}{c}}\le 9e$$

Equality holds for $a=b=c=1$.

6.44 If $a,b,c,d,x,y,z,t>0$ **then:**

$$\frac{(ax)^a\cdot(by)^b\cdot(cz)^c\cdot(dt)^d}{(a+b+c+d)^{a+b+c+d}}\ge\left(\frac{xyzt}{xyz+xyt+xzt+yzt}\right)^{a+b+c+d}$$

Daniel Sitaru

Solution

If $a,b,c,d,u,w,s>0$ *by weighted GM-HM:*

$$\sqrt[a+b+c]{\left(\frac{a}{u}\right)^a\cdot\left(\frac{b}{v}\right)^b\cdot\left(\frac{c}{w}\right)^c\cdot\left(\frac{d}{s}\right)^d}\ge\frac{a+b+c+d}{\frac{a}{\frac{a}{u}}+\frac{b}{\frac{b}{v}}+\frac{c}{\frac{c}{w}}+\frac{d}{\frac{d}{s}}}=\frac{a+b+c+d}{u+v+w+s}$$

$$\left(\frac{a}{u}\right)^a\cdot\left(\frac{b}{v}\right)^b\cdot\left(\frac{c}{w}\right)^c\cdot\left(\frac{d}{s}\right)^d\ge\left(\frac{a+b+c+d}{u+v+w+s}\right)^{a+b+c+d}$$

Let be $u=\frac{1}{x};v=\frac{1}{y};w=\frac{1}{z};s=\frac{1}{t}$

$$(ax)^a\cdot(by)^b\cdot(cz)^c\cdot(dt)^d\ge\frac{(a+b+c+d)^{a+b+c+d}}{\left(\frac{1}{x}+\frac{1}{y}+\frac{1}{z}+\frac{1}{t}\right)^{a+b+c+d}}$$

$$\frac{(ax)^a\cdot(by)^b\cdot(cz)^c\cdot(dt)^d}{(a+b+c+d)^{a+b+c+d}}\ge\left(\frac{1}{\frac{xyz+xyt+xzt+yzt}{xyzt}}\right)^{a+b+c+d}=$$

$$=\left(\frac{xyzt}{xyz+xyt+xzt+yzt}\right)^{a+b+c+d}$$

6.45 If $a,b,c,x,y,z>0$; $a+b+c\ge x+y+z$ then:

$$a^ab^bc^c\ge x^ay^bz^c$$

Daniel Sitaru

Solution

By weighted GM-HM:

$$\sqrt[a+b+c]{\left(\frac{a}{x}\right)^a\cdot\left(\frac{b}{y}\right)^b\cdot\left(\frac{c}{z}\right)^c}\ge\frac{a+b+c}{\left(\frac{a}{\frac{a}{x}}\right)+\left(\frac{b}{\frac{b}{y}}\right)+\left(\frac{c}{\frac{c}{z}}\right)}=\frac{a+b+c}{x+y+z}\ge$$

$$\ge\frac{x+y+z}{x+y+z}\ge 1;\quad \sqrt[a+b+c]{\left(\frac{a}{x}\right)^a\cdot\left(\frac{b}{y}\right)^b\cdot\left(\frac{c}{z}\right)^c}\ge 1$$

$$\left(\frac{a}{x}\right)^a \cdot \left(\frac{b}{y}\right)^b \cdot \left(\frac{c}{z}\right)^c \geq 1;\ \ a^a \cdot b^b \cdot c^c \geq x^a \cdot y^b \cdot z^c$$

Equality holds for $x = a; y = b; z = c$.

6.46 If $a, b, c, x, y, z > 0; a + b + c = 3$ then:

$$\boldsymbol{a^a \cdot b^b \cdot c^c (x + y + z)^3 \geq 27 x^a y^b z^c}$$

When does the equality holds?

Daniel Sitaru

Solution

By weighted GM-HM:

$$\sqrt[a+b+c]{\left(\frac{a}{x}\right)^a \cdot \left(\frac{b}{y}\right)^b \cdot \left(\frac{c}{z}\right)^c} \geq \frac{a+b+c}{\left(\frac{a}{\frac{a}{x}}\right) + \left(\frac{b}{\frac{b}{y}}\right) + \left(\frac{c}{\frac{c}{z}}\right)} = \frac{a+b+c}{x+y+z}$$

$$\left(\frac{a}{x}\right)^a \cdot \left(\frac{b}{y}\right)^b \cdot \left(\frac{c}{z}\right)^c \geq \left(\frac{a+b+c}{x+y+z}\right)^{a+b+c} = \left(\frac{3}{x+y+z}\right)^3$$

$$\frac{a^a}{x^a} \cdot \frac{b^b}{y^b} \cdot \frac{c^c}{z^c} \geq \frac{27}{(x+y+z)^3}$$

$$a^a \cdot b^b \cdot c^c (x + y + z)^3 \geq 27 x^a y^b z^c$$

Equality holds for $a = b = c = 1$ *and* $x = y = z$.

6.47 If $x > 0$ then:

$$\boldsymbol{x^2 - 3x + 1 \geq log x^x - x^{x+1}}$$

Lazaros Zacharidis

Solution (Michael Sterghiou)

$$x^2 - 3x + 1 \geq log x^x - x^{x+1} \quad (1)$$

$x^2 + 1 \overset{Am-Gm}{\geq} 2x$ *so (1) becomes the stronger inequality*

$$-x - x log x + x^{x+1} \geq 0 \text{ or } x > 0 \Rightarrow -1 - log x + x^x \geq 0$$

Consider the function $f(x) = x^x - logx - 1, f^{'}(x) = x^x(1 + logx) - \frac{1}{x}$

If $x < 1$ *then* $x^x < 1; 1 + logx < 1; \frac{1}{x} > 1 \Rightarrow f^{'}(x) < 0$

If $x > 1$ *then* $f^{'}(x) > 0, f^{'}(1) = 0.$

Therefore $f(x) \uparrow, x \in [1, \infty); f(x) \downarrow x \in (0,1]$ *hence:* $f(1) = 0$ *is a global min on* $(0, \infty)$ *because* $\lim_{x\to 0_+} f(x) = +\infty;\ \lim_{x\to\infty} f(x) = +\infty.$

Now: $f(x) \geq f(0) = 0.$

6.48 $\boldsymbol{x, y, z}$ **– real numbers different in pairs** $\boldsymbol{x + y + z = 0.}$

Find min $\boldsymbol{\Omega}$

$$\boldsymbol{\Omega = (x^2 + y^2 + z^2)\left(\frac{1}{(x-y)^2} + \frac{1}{(y-z)^2} + \frac{1}{(z-x)^2}\right)}$$

Le Ngo Duc

Solution (Tran Hong)

$x + y + z = 0 \Leftrightarrow z = -x - y; (x \neq y; x \neq z; y \neq z)$. *We must show that:*

$$\Omega \geq \frac{9}{2}$$

$$\Leftrightarrow [x^2 + y^2 + (x+y)^2]\left[\frac{1}{(x-y)^2} + \frac{1}{(2y+x)^2} + \frac{1}{(x+2y)^2}\right] \geq \frac{9}{2}$$

$$\Leftrightarrow 2[2x^2 + 2y^2 + 2xy][(2y+x)^2(x+2y)^2 + (x-y)^2(x+2y)^2 + (x-y)^2(y+2x)^2] \geq$$

$$\geq 9\{(x-y)(x+2y)(y+2x)\}^2 \Leftrightarrow 243x^4y^2 + 486x^3y^3 + 243x^2y^4 \geq 0$$

$$\Leftrightarrow 243(x^2y^2)(x^2 + 2xy + y^2) \geq 0 \Leftrightarrow 243(xy)^2(x+y)^2 \geq 0 \text{ (true for } x, y)$$

Equality:

$x = 0 \Rightarrow z = -y \neq 0;\ y = 0 \Leftrightarrow z = -x \neq 0;\ x = -y(x, y \neq 0) \Rightarrow$
$z = -(-y) - y = 0$

Hence, $\Omega_{\min} = \frac{9}{2} \Leftrightarrow (x; y; z) = (0; t; -t)$ *or* $(x; y; z) = (t; -t; 0)$ *or* $(t; 0; -t)$ *(with* $t \neq 0$*)*

6.49 $\boldsymbol{n} \in \mathbb{R}, \boldsymbol{n} > 1, \boldsymbol{n}$ **– fixed,** $\boldsymbol{\Omega_n(x,y)} = \sqrt{\frac{x}{x+n^2-1}} + \sqrt{\frac{y}{y+n^2-1}}$

Find:

$$\boldsymbol{\Omega_1} = \min_{\substack{x,y\geq 0\\ x+y=2}} \boldsymbol{\Omega_n(x,y)}, \boldsymbol{\Omega_2} = \max_{\substack{x,y\geq 0\\ x+y=2}} \boldsymbol{\Omega_n(x,y)}$$

Marin Chirciu

Solution (Adrian Popa)

$$x, y > 0$$

$$x + y = 2 \Rightarrow x = 2 - y$$

$$n \geq 1 \Rightarrow \begin{matrix} n^2 - 1 > 0 \\ x > 0 \end{matrix} \Rightarrow x + n^2 - 1 \neq 0$$

$$n > 1 \Rightarrow \left.\begin{matrix} n^2 - 1 > 0 \\ y > 0 \end{matrix}\right\} \Rightarrow y + n^2 - 1 \neq 0$$

$$\Omega_n(x) = \sqrt{\frac{x}{x + n^2 - 1}} + \sqrt{\frac{2 - x}{-x + n^2 + 1}}$$

$$\Omega_n'(x) = \frac{\frac{x + n^2 - 1 - x}{(x + n^2 - 1)^2}}{2\sqrt{\frac{x}{x + n^2 - 1}}} + \frac{\frac{x - n^2 - 1 + 2 - x}{(-x + n^2 + 1)^2}}{2\sqrt{\frac{2 - x}{-x + n^2 + 1}}} =$$

$$= \frac{(n^2 - 1)\sqrt{x + n^2 - 1}}{2\sqrt{x}(x + n^2 - 1)^2} - \frac{(n^2 - 1)\sqrt{-x + n^2 + 1}}{2\sqrt{2 - x}(-x + n^2 + 1)^2} = 0 \left| : \frac{n^2 - 1}{2} \right. \Rightarrow$$

$$\Rightarrow \frac{\sqrt{x + n^2 - 1}}{\sqrt{x}(x + n^2 - 1)^2} = \frac{\sqrt{-x + n^2 + 1}}{\sqrt{2 - x}(-x + n^2 + 1)^2}\Bigg|^2 \Rightarrow \frac{x + n^2 - 1}{x(x + n^2 - 1)^4} =$$

$$= \frac{-x + n^2 + 1}{(2 - x)(-x + n^2 + 1)^4} \Rightarrow x(x + n^2 - 1)^3 = (2 - x)(-x + n^2 + 1)^3$$

We notice that: If $x > 1 \Rightarrow x > 2 - x \Rightarrow 2x > 2 \Rightarrow x > 1$ *(True)*

$x + n^2 - 1 > -x + n^2 + 1 \Rightarrow 2x > 2 \Rightarrow x > 1$ *(True)*

$$\Rightarrow x(x^2 + n - 1)^3 > (2 - x)(-x + n^2 + 1)^3$$

If $x < 1 \Rightarrow x < 2 - x \Rightarrow 2x < 2 \Rightarrow x < 1$ *(True)*

$$x+n^2-1<-x+n^2+1 \Rightarrow 2x<2 \Rightarrow x<1 \quad \textit{(True)}$$

$$\Rightarrow x(x^2+n-1)^3<(2-x)(-x+n^2+1)^3$$

So, the only solution $\Omega_n'(x)=0$ *is* $x=1$

x	0	1	2
$\Omega_n'(x)$	$+ + + + + + + + +$	0	$- - - - - - - - -$
$\Omega_n(x)$	$\sqrt{\frac{2}{n^2+1}}$ ↗	$\frac{2}{n}$	↘ $\sqrt{\frac{2}{n^2+1}}$

$$\Omega_n(0)=\Omega_n(2)=\sqrt{\frac{2}{n^2+1}}, \Omega_n(1)=\frac{1}{n}+\frac{1}{n}=\frac{2}{n}$$

$$\textit{So, } \Omega_1=\sqrt{\frac{2}{n^2+1}} \textit{ and } \Omega_2=\frac{2}{n}.$$

6.50 If $x,y,z>0, x\geq y+z$ then find the minimum value of:

$$(a)\ \frac{y+z}{x}+\frac{z+x}{y}+\frac{x+y}{z}$$

$$(b)\ \frac{x}{y+z}+\frac{y}{z+x}+\frac{z}{x+y}$$

Hung Nguyen Viet

Solution (Tran Hong)

a) For $x,y,z>0$ and $x\geq y+z$: Let $f(x)=\frac{y+z}{x}+\frac{z+x}{y}+\frac{x+y}{z}$

$$\rightarrow f'(x)=-\frac{y+z}{x^2}+\frac{1}{y}+\frac{1}{z} \overset{\forall x\geq y+z}{\geq} -\frac{1}{y+z}+\frac{1}{y}+\frac{1}{z}=\frac{y+z}{yz}-\frac{1}{y+z}$$

$$=\frac{(y+z)^2-yz}{y+z}=\frac{y^2+z^2+yz}{yz}>0\ (\because y,z>0)$$

$$\rightarrow f(x)\uparrow [y+z\,;+\infty) \rightarrow f(x)\geq f(y+z)=1+\frac{y+2z}{y}+\frac{2y+z}{2}$$

$$=3+2\left(\frac{y}{z}+\frac{z}{y}\right) \overset{AM-GM}{\geq} 3+2.2=7;$$

$$\to \Omega = \frac{y+z}{x} + \frac{z+x}{y} + \frac{x+y}{z} \geq 7 \to \Omega_{min} = 7 \leftrightarrow \begin{cases} x = y+z \\ y = z \end{cases} \leftrightarrow \begin{cases} x = 2z \\ y = z \end{cases}$$

b) For $x, y, z > 0$ and $x \geq y+z$:

Let $g(x) = \frac{x}{y+z} + \frac{y}{z+x} + \frac{z}{x+y} \to g'(x) = \frac{1}{y+z} - \frac{y}{(z+x)^2} - \frac{z}{(x+y)^2}$

$$\to g''(x) = \frac{2y}{(z+x)^3} + \frac{2z}{(x+y)^3} > 0 \quad (\therefore \forall x \geq y+z > 0)$$

$$\to g'(x) \uparrow [y+z\,;+\infty) \to g'(x) \geq g'(y+z)$$

$$= \frac{1}{y+z} - \frac{y}{(y+2z)^2} - \frac{z}{(2y+z)^2}$$

$$> \frac{1}{y+z} - \frac{y}{(y+z)^2} - \frac{z}{(y+z)^2} = \frac{1}{y+z} - \frac{y+z}{(y+z)^2} = \frac{1}{y+z} - \frac{1}{y+z} = 0;$$

$$(\therefore \forall x \geq y+z > 0)$$

$$\to g(x) \uparrow [y+z\,;+\infty) \to g(x) \geq g(y+z) = 1 + \frac{y}{y+2z} + \frac{z}{2y+z} \overset{(*)}{\geq} \frac{5}{3}$$

$$(*) \leftrightarrow \frac{y}{y+2z} + \frac{z}{2y+z} \geq \frac{2}{3}$$

$$\leftrightarrow 3[y(2y+z) + z(y+2z)] \geq 2(y+2z)(z+2y)$$

$$\leftrightarrow 3(2y^2 + 2z^2 + 2yz) \geq 2(2y^2 + 2z^2 + 5yz)$$

$$\leftrightarrow 2y^2 + 2z^2 - 4yz \geq 0 \leftrightarrow 2(y-z)^2 \geq 0 \ (\therefore true)$$

$$\to \mho = \frac{x}{y+z} + \frac{y}{z+x} + \frac{z}{x+y} \geq \frac{5}{3} \to \mho_{min} = \frac{5}{3} \leftrightarrow \begin{cases} x = y+z \\ y = z \end{cases} \leftrightarrow \begin{cases} x = 2z \\ y = z \end{cases}$$

6.51 If $a, b, c, d > 0, ac = bd$ then:

$$\frac{(a+b)(b+c)(c+d)(d+a)}{ab(c+d) + cd(a+b)} \geq 4\sqrt{ac}$$

Daniel Sitaru

Solution(Marian Ursărescu)

Because $d = \frac{ac}{b} \Rightarrow$ *we must show:* $\frac{(a+b)(b+c)\left(c+\frac{ac}{b}\right)\left(\frac{ac}{b}+a\right)}{ab\left(c+\frac{ac}{b}\right)+c\cdot\frac{ac}{b}(a+b)} \geq 4\sqrt{ac} \Leftrightarrow$

$$\Leftrightarrow \frac{\frac{(a+b)(b+c)c(b+a)\cdot a(c+b)}{b^2}}{abc\frac{(a+b)}{b}+\frac{ac^2(a+b)}{b}} \geq 4\sqrt{ac} \Leftrightarrow \frac{(a+b)^2(b+c)^2ac}{ac(a+b)(b+c)\cdot b} \geq 4\sqrt{ac} \Leftrightarrow$$

$$(a+b)(b+c) \geq 4b\sqrt{ac} \quad \textit{(1)}$$

(1) it is true because: $\left.\begin{matrix} a+b \geq 2\sqrt{ab} \\ b+c \geq 2\sqrt{bc} \end{matrix}\right\} \Rightarrow (a+b)(b+c) \geq 4b\sqrt{ac}$

6.52 If $a, b, c > 1, abc = e^3$ then:

$$\left(\log\sqrt{ab}\right)^{\log\sqrt{ab}} \cdot \left(\log\sqrt{bc}\right)^{\log\sqrt{bc}} \cdot \left(\log\sqrt{ca}\right)^{\log\sqrt{ca}} \geq \log a \cdot \log b \cdot \log c$$

Daniel Sitaru

Solution (Marian Ursărescu)

$$\left(\ln\sqrt{ab}\right)^{\ln\sqrt{ab}}\left(\ln\sqrt{bc}\right)^{\ln\sqrt{bc}}\left(\ln\sqrt{ca}\right)^{\ln\sqrt{ca}} \geq \ln a \ln b \ln c$$

We must show: $\left(\frac{\ln a+\ln b}{2}\right)^{\frac{\ln a+\ln b}{2}}\left(\frac{\ln b+\ln c}{2}\right)^{\frac{\ln b+\ln c}{2}}\left(\frac{\ln c+\ln a}{2}\right)^{\frac{\ln c+\ln a}{2}} \geq$

$$\ln a \ln b \ln c \quad \textit{(1)}$$

Let $\ln a = x, \ln b = y, \ln c = z, x, y, z > 0$ *(2)*

and $abc = e^3 \Leftrightarrow x+y+z = 3$. *From (1)+(2) we must show:*

$$\left(\frac{x+y}{2}\right)^{\frac{x+y}{2}} \cdot \left(\frac{y+z}{2}\right)^{\frac{y+z}{2}} \cdot \left(\frac{z+x}{2}\right)^{\frac{z+x}{2}} \geq xyz \Leftrightarrow$$

$$\ln\left[\left(\frac{x+y}{2}\right)^{\frac{x+y}{2}}\left(\frac{y+z}{2}\right)^{\frac{y+z}{2}} \cdot \left(\frac{z+x}{2}\right)^{\frac{z+x}{2}}\right] \geq \ln(xyz) \Leftrightarrow$$

$$\sum \frac{x+y}{2}\ln\left(\frac{x+y}{2}\right) \geq \ln x + \ln y + \ln z \quad \textit{(3)}$$

But $\frac{x+y}{2} \geq \sqrt{xy} \Rightarrow \ln\left(\frac{x+y}{2}\right) \geq \ln\sqrt{xy} = \frac{1}{2}(\ln x + \ln y)$ *(4)*

From (3)+(4) we must show: $\sum \frac{x+y}{4}(\ln x + \ln y) \geq \ln x + \ln y + \ln z \Leftrightarrow$

$$\left.\begin{matrix} \sum(2x+y+z)\ln x \geq 4(\ln x + \ln y + \ln z) \\ But\ x+y+z = 3 \end{matrix}\right\} \Rightarrow$$

$$\sum(x+3)\ln x \geq 4(\ln x+\ln y+\ln z) \Leftrightarrow$$

$$\sum(x-1)\ln x \geq 0, \textit{ true because:}$$

$$\textit{if } x>1 \Rightarrow \ln x>0 \textit{ and } x\in(0,1) \Rightarrow \ln x<0$$

$$\textit{equality for } x=y=z=1.$$

6.53 If $x>0$ then:

$$\{x\}+\left\{x+\frac{1}{2}\right\}+\left\{2x+\frac{1}{2}\right\}>2\sqrt{\{4x\}};$$

$$\{x\}=x-[x];\ [*] \textbf{ - great integer function}$$

Daniel Sitaru

Solution

$$\{2x\}-\{x\}-\left\{x+\frac{1}{2}\right\}=$$

$$=2x-[2x]-(x-[x])-\left(x+\frac{1}{2}-\left[x+\frac{1}{2}\right]\right)=$$

$$=2x-[2x]-x+[x]-x-\frac{1}{2}+\left[x+\frac{1}{2}\right]=$$

$$=[x]+\left[x+\frac{1}{2}\right]-[2x]-\frac{1}{2}\overset{HERMITE}{=}[2x]-[2x]-\frac{1}{2}=-\frac{1}{2}$$

$$\{2x\}-\{x\}-\left\{x+\tfrac{1}{2}\right\}=-\tfrac{1}{2}\quad (1)$$

Replacing x *with* $2x$ *in (1):* $\{4x\}-\{2x\}-\left\{2x+\tfrac{1}{2}\right\}=-\tfrac{1}{2}\quad$ *(2)*

By adding (1); (2): $\{4x\}-\{x\}-\left\{x+\tfrac{1}{2}\right\}-\left\{2x+\tfrac{1}{2}\right\}=-1$

$$\{x\}+\left\{x+\frac{1}{2}\right\}+\left\{2x+\frac{1}{2}\right\}=1+\{4x\}\overset{AM-GM}{>}2\sqrt{1\cdot\{4x\}}=2\sqrt{\{4x\}}$$

6.54 If $0<x,y,z$ then:

$$x(y^2[x]+z^2\{x\})\geq(y[x]+z\{x\})^2,$$

$$\{x\}=x-[x],\ [*] \textbf{ −great integer function.}$$

Daniel Sitaru

Solution(Adrian Popa)

$$([x]+\{x\})(y^2[x]+z^2\{x\}) = y^2[x]^2 + z^2[x]\{x\} + y^2[x]\{x\} + z^2\{x\}^2$$
$$= y^2[x]^2 + z^2\{x\}^2 + [x]\{x\}\underbrace{(z^2+y^2)}_{2yz} \geq y^2[x]^2 + z^2\{x\}^2 + 2yz[x]\{x\}$$
$$= (y[x]+z\{x\})^2$$

6.55 Prove:

$$\frac{\tan^{-1}x}{x} + \frac{\tan^{-1}y}{y} \geq \frac{\pi}{2}; 0 < x, y \leq 1$$

Jalil Hajimir

Solution (Daniel Sitaru)

$$f:(0,1] \to \mathbb{R}, f(x) = \frac{\tan^{-1}x}{x}, f'(x) = \frac{\frac{x}{1+x^2} - \tan^{-1}x}{x^2}$$
$$g:(0,1] \to \mathbb{R}, g(x) = \frac{x}{1+x^2} - \tan^{-1}x, g'(x) = \frac{-2x^2}{(1+x^2)^2} < 0$$
$$\sup g(x) = \lim_{\substack{x\to 0\\ x>0}} g(x) = 0 \Rightarrow g(x) < 0 \Rightarrow f'(x) < 0$$
$$f - decreasing, \min f(x) = f(1) = \tfrac{\pi}{4} \Rightarrow f(x) \geq \tfrac{\pi}{4}$$
$$\frac{\tan^{-1}x}{x} + \frac{\tan^{-1}y}{y} = f(x) + f(y) \geq \frac{\pi}{4} + \frac{\pi}{4} = \frac{\pi}{2}$$

Equality holds for $x = y = 1$

6.56 If $x, y, z > 0, x + y + z = 6$ then:

$$\Gamma(x) + \Gamma(y) + \Gamma(z) \geq 3$$

Jalil Hajimir

Solution (Daniel Sitaru)

$$\int_0^\omega te^{-t}dt = \frac{-\omega}{e^\omega} - \frac{1}{e^\omega} + 1, \lim_{\omega\to\infty}\int_0^\omega te^{-t}dt = \lim_{\omega\to\infty}\left(\frac{-\omega}{e^\omega} - \frac{1}{e^\omega} + 1\right) = 1$$

$$e^{-t}t^{x-1} + e^{-t}t^{y-1} + e^{-t}t^{z-1} \overset{AM-GM}{\geq} 3e^{-t}\sqrt[3]{t^{x-1}\cdot t^{y-1}\cdot t^{z-1}} =$$

$$= 3e^{-t}\sqrt[3]{t^{x+y+z-3}} = 3e^{-t}\sqrt[3]{t^{6-3}} = 3te^{-t}$$

$$\Gamma(x) + \Gamma(y) + \Gamma(z) \geq 3 \lim_{\omega\to\infty} \int_0^{\omega} te^{-t}dt = 3$$

Equality holds for $x = y = z = 2$.

6.57 If $a, b, c \in \mathbb{N}$; $ab + bc + ca = 27$ then:

$$135 + \sqrt[a+b]{a^{2a} \cdot b^{2b}} + \sqrt[b+c]{b^{2b} \cdot c^{2c}} + \sqrt[c+a]{c^{2c} \cdot a^{2a}} \leq 2(a+b+c)^2$$

Daniel Sitaru

Solution

$$\sqrt[a+b]{a^{2a} \cdot b^{2b}} = \sqrt[a+b]{(a^2)^a \cdot (b^2)^b} \overset{AM-GM}{\leq} \frac{\underbrace{a^2 + a^2 + \dots + a^2}_{for\ "a"\ times} + \underbrace{b^2 + b^2 + \dots + b^2}_{for\ "b"\ times}}{a+b} =$$

$$= \frac{a \cdot a^2 + b \cdot b^2}{a+b} = \frac{a^3 + b^3}{a+b} = \frac{(a+b)(a^2 - ab + b^2)}{a+b} = a^2 - ab + b^2$$

$$\sum_{cyc} \sqrt[a+b]{a^{2a} \cdot b^{2b}} \leq \sum_{cyc} (a^2 - ab + b^2) =$$

$$= 2\sum_{cyc} a^2 - \sum_{cyc} ab = 2(a+b+c)^2 - 4\sum_{cyc} ab - \sum_{cyc} ab =$$

$$= 2(a+b+c)^2 - 5\sum_{cyc} ab = 2(a+b+c)^2 - 135$$

$$135 + \sum_{cyc} \sqrt[a+b]{a^{2a} \cdot b^{2b}} \leq 2(a+b+c)^2$$

Equality holds for $a = b = c = 3$.

6.58 If $a, b, c \geq 0$ then:

$$\left(\frac{ab}{1+a} + \frac{bc+a}{(1+a)(1+b)}\right)^6 \leq \frac{ab^6}{1+a} + \frac{bc^6 + a^6}{(1+a)(1+b)}$$

Daniel Sitaru

Solution

$$\frac{a}{1+a}+\frac{b}{(1+a)(1+b)}+\frac{1}{(1+a)(1+b)}=$$

$$=\frac{1+a-1}{1+a}+\frac{1+b-1}{(1+a)(1+b)}+\frac{1}{(1+a)(1+b)}=$$

$$=\left(1-\frac{1}{1+a}\right)+\left(\frac{1}{1+a}-\frac{1}{(1+a)(1+b)}\right)+\frac{1}{(1+a)(1+b)}=1$$

Let be $f:[0,\infty)\to\mathbb{R}; f(x)=x^6, f'(x)=6x^5; f''(x)=30x^4\geq 0$

By Jensen's inequality for $\lambda_1,\lambda_2,\lambda_3>0$*; then:* $\lambda_1+\lambda_2+\lambda_3=1$

$$f(\lambda_1x_1+\lambda_2x_2+\lambda_3x_3)\leq\lambda_1f(x_1)+\lambda_2f(x_2)+\lambda_3f(x_3)$$

For $\lambda_1=\frac{a}{1+a};\ \lambda_2=\frac{b}{(1+a)(1+b)};\ \lambda_3=\frac{1}{(1+a)(1+b)}$

$$f\left(\frac{a}{1+a}x_1+\frac{b}{(1+a)(1+b)}x_2+\frac{1}{(1+a)(1+b)}x_3\right)\leq$$

$$\leq\frac{a}{1+a}f(x_1)+\frac{b}{(1+a)(1+b)}f(x_2)+\frac{1}{(1+a)(1+b)}f(x_3)$$

$$\left(\frac{a}{1+a}x_1+\frac{b}{(1+a)(1+b)}x_2+\frac{1}{(1+a)(1+b)}x_3\right)^6\leq$$

$$\leq\frac{a}{1+a}x_1^6+\frac{b}{(1+a)(1+b)}x_2^6+\frac{1}{(1+a)(1+b)}x_3^6$$

For $x_1=b; x_2=c; x_3=a$

$$\left(\frac{ab}{1+a}+\frac{bc}{(1+a)(1+b)}+\frac{a}{(1+a)(1+b)}\right)^6\leq\frac{ab^6}{1+a}+\frac{ba^6+c^6}{(1+a)(1+b)}$$

$$\left(\frac{ab}{1+a}+\frac{bc+a}{(1+a)(1+b)}\right)^6\leq\frac{ab^6}{1+a}+\frac{ba^6+c^6}{(1+a)(1+b)}$$

6.59 If $a, b, c, x_1, x_2, x_3 > 0$ then:

$$(a+b+c)^5\left(ax_1^7+bx_2^7+cx_3^7\right) \geq (ax_1+bx_2+cx_3)^5\left(ax_1^2+bx_2^2+cx_3^2\right)$$

and prove that:

$$(a+b+c)^5\left(\frac{a}{b^7}+\frac{b}{c^7}+\frac{c}{a^7}\right) \geq \left(\frac{a}{b}+\frac{b}{c}+\frac{c}{a}\right)^5\left(\frac{a}{b^2}+\frac{b}{c^2}+\frac{c}{a^2}\right)$$

Daniel Sitaru

Solution

$$ax_1^7+bx_2^7+cx_3^7 = \frac{a(x_1^2)^6}{x_1^5}+\frac{b(x_2^2)^6}{x_2^5}+\frac{c(x_2^2)^6}{x_3^5} =$$

$$= \frac{\left(ax_1^2\right)^6}{(ax_1)^5}+\frac{\left(bx_2^2\right)^6}{(bx_2)^5}+\frac{\left(cx_3^2\right)^6}{(cx_3)^5} \overset{RADON}{\geq} \frac{\left(ax_1^2+bx_2^2+cx_3^2\right)^6}{(ax_1+bx_2+cx_3)^5} \quad (1)$$

$$ax_1^2+bx_2^2+cx_3^2 = \frac{(ax_1)^2}{a}+\frac{(bx_2)^2}{b}+\frac{(cx_3)^2}{c} \overset{RADON}{\geq} \frac{(ax_1+bx_2+cx_3)^2}{a+b+c}$$

$$a+b+c \geq \frac{(ax_1+bx_2+cx_3)^2}{ax_1^2+bx_2^2+cx_3^2}$$

$$(a+b+c)^5 \geq \frac{(ax_1+bx_2+cx_3)^{10}}{\left(ax_1^2+bx_2^2+cx_3^2\right)^5} \quad (2)$$

By multiplying (1); (2):

$$(a+b+c)^5(ax_1^7+bx_2^7+cx_3^7) \geq$$

$$\geq \frac{(ax_1^2+bx_2^2+cx_3^2)^6}{(ax_1+bx_2+cx_3)^5}\cdot\frac{(ax_1+bx_2+cx_3)^{10}}{(ax_1^2+bx_2^2+cx_3^2)^5}$$

$$(a+b+c)^5(ax_1^7+bx_2^7+cx_3^7) \geq (ax_1+bx_2+cx_3)^5(ax_1^2+bx_2^2+cx_3^2)$$

For $x_1 = \frac{1}{b}; x_2 = \frac{1}{c}; x_3 = \frac{1}{a}$

$$(a+b+c)^5\left(\frac{a}{b^7}+\frac{b}{c^7}+\frac{c}{a^7}\right) \geq \left(\frac{a}{b}+\frac{b}{c}+\frac{c}{a}\right)^5\left(\frac{a}{b^2}+\frac{b}{c^2}+\frac{c}{a^2}\right)$$

6.60 If $a_n = 1 + \frac{1}{\sqrt{2}} + \frac{1}{\sqrt{3}} + \cdots + \frac{1}{\sqrt{n}}, n \geq 1$ then:

$$\frac{1}{a_1^2} + \frac{1}{a_2^2\sqrt{2}} + \frac{1}{a_3^2\sqrt{3}} + \cdots + \frac{1}{a_n^2\sqrt{n}} < 2 - \frac{1}{2\sqrt{n}}, n \geq 1$$

Ionuţ Florin Voinea

Solution (Catinca Alexandru)

$$a_n = 1 + \frac{1}{\sqrt{2}} + \frac{1}{\sqrt{3}} + \cdots + \frac{1}{\sqrt{n}}, n \geq 1$$

$$\frac{1}{a_1^2} + \frac{1}{a_2^2\sqrt{2}} + \frac{1}{a_3^2\sqrt{3}} + \cdots + \frac{1}{a_n^2\sqrt{n}} < 2 - \frac{1}{2\sqrt{n}}, n \geq 1; \quad (1)$$

$$a_n - a_{n-1} = \frac{1}{\sqrt{n}} > 0 \Rightarrow a_n \geq a_{n-1} \Rightarrow a_n^2 \geq a_n \cdot a_{n-1}$$

$$\Rightarrow S = \frac{1}{a_1^2} + \frac{1}{a_2^2\sqrt{2}} + \frac{1}{a_3^2\sqrt{3}} + \cdots + \frac{1}{a_n^2\sqrt{n}}$$

$$< 1 + \frac{1}{\sqrt{2} \cdot a_1 a_2} + \frac{1}{\sqrt{3} \cdot a_2 a_3} + \cdots + \frac{1}{\sqrt{n} \cdot a_{n-1} a_n}$$

$$= 1 + (a_2 - a_1)\frac{1}{a_1 a_2} + (a_3 - a_2)\frac{1}{a_2 a_3} + \cdots + (a_n - a_{n-1})\frac{1}{a_{n-1} a_n}$$

$$= 1 + \frac{1}{a_1} - \frac{1}{a_1} + \frac{1}{a_2} - \frac{1}{a_2} + \cdots + \frac{1}{a_{n-1}} - \frac{1}{a_n} = 2 - \frac{1}{a_n}$$

We must show: $2 - \frac{1}{a_n} < 2 - \frac{1}{2\sqrt{n}} \Leftrightarrow a_n < 2\sqrt{n}; \quad (*)$

Prove $a_n < 2\sqrt{n}$ *by induction.* $P_1: a_1 = 1 < 2$ *true.* $P_n \to P_{n+1}$

$$a_n < 2\sqrt{n} \Rightarrow a_n + \frac{1}{\sqrt{n+1}} < 2\sqrt{n} + \frac{1}{\sqrt{n+1}} \Leftrightarrow a_{n+1} < 2\sqrt{n} + \frac{1}{\sqrt{n+1}}$$

$$2\sqrt{n} + \frac{1}{\sqrt{n+1}} \overset{?}{<} 2\sqrt{n+1}\Big|^2 \Leftrightarrow 4n + 4\sqrt{\frac{n}{n+1}} + \frac{1}{n+1} < 4n + 4$$

$$\Leftrightarrow 1 < \frac{n}{n+1} - 4\sqrt{\frac{n}{n+1}} + 4 \Leftrightarrow 1 < \left(\sqrt{\frac{n}{n+1}} - 2\right)^2$$

$$\Leftrightarrow 1 < \left|\sqrt{\frac{n}{n+1}} - 2\right| \overset{n \in \mathbb{N}}{\Longleftrightarrow} 1 > \sqrt{\frac{n}{n+1}} \text{ true, then } a_{n+1} < 2\sqrt{n+1}$$

So, $a_n < 2\sqrt{n} \Rightarrow (*)true \Rightarrow (1)$ *is true.*

6.61 Find the maximum of $z = \frac{2x}{y+5} + \frac{3y}{x+2}, 0 \le x, y \le 4$

Jalil Hajimir

Solution(Daniel Sitaru)

$$z: [0,4]X[0,4] \to \mathbb{R}, z(x,y) = \frac{2x}{y+5} + \frac{3y}{x+2}$$

$$z_x^{'} = \frac{2}{y+5} - \frac{3y}{(x+2)^2}, z_{xx}^{''} = \frac{6y}{(x+2)^3} \ge 0$$

$$z_y^{'} = \frac{-2x}{(y+5)^2} + \frac{3}{(x+2)^2}, z_{yy}^{''} = \frac{4x}{(y+5)^3} \ge 0$$

$[0,4]X[0,4]$ *—compact domain. By Weierstrass z-is bounded and attains max.*

$[0,4]X[0,4]$ *—convexe domain with borders parallel with axis, z —convexe in each variable. By Gireaux max is attained in one of the vertexes:*

$$maxz = max\{z(0,0), z(0,4), z(4,0), z(4,4)\} = max\left\{0, 6, \frac{8}{5}, \frac{26}{9}\right\} = 6$$

6.62 If $m, n \in \mathbb{N} - \{0\}, F_n$ —Fibonacci numbers, L_n —Lucas numbers then:

$$\sqrt[5]{\frac{F_m^2 F_n^3 L_n^2 L_m^3}{F_{m+n}^5}} + \sqrt[5]{\frac{F_m^3 F_n^2 L_n^3 L_m^2}{F_{m+n}^5}} < 2$$

Daniel Sitaru

Solution (Marian Ursărescu)

We must show: $\sqrt[5]{F_m^2 F_n^3 L_n^2 L_m^3} + \sqrt[5]{F_m^3 F_n^2 L_n^3 L_m^2} < 2F_{m+n}; \ (1)$

$$\sqrt[5]{F_m^2 F_n^3 L_n^2 L_m^3} = \sqrt[5]{(F_m \cdot L_n)^2 \cdot (F_n \cdot L_m)^3} \overset{Am-Gm}{\le} \frac{2F_m \cdot L_n + 3F_n \cdot L_m}{5}; \ (2)$$

$$\sqrt[5]{F_m^3 F_n^2 L_n^3 L_m^2} = \sqrt[5]{(F_m \cdot L_n)^3 \cdot (F_n \cdot L_m)^2} \overset{Am-Gm}{\le} \frac{3F_m \cdot L_n + 2F_n \cdot L_m}{5}; \ (3)$$

From (2),(3) we have: $\sqrt[5]{F_m^2 F_n^3 L_n^2 L_m^3} + \sqrt[5]{F_m^3 F_n^2 L_n^3 L_m^2} < F_m \cdot L_n + F_n \cdot L_m; (4)$

But: $F_m \cdot L_n + F_n \cdot L_m = 2F_{m+n}$ *(Ferns identity); (5)*

From (4),(5) we get (1) is true.

6.63 If $x, y, z, t \in (0,1)$; $3\sqrt{3}(xyz + yzt + ztx + txy) = 4$ then:

$$\frac{yzt}{x(1-x^2)} + \frac{ztx}{y(1-y^2)} + \frac{txy}{z(1-z^2)} + \frac{xyz}{t(1-t^2)} \ge 2$$

Daniel Sitaru

Solution

$$2x^2 \cdot (1-x^2)(1-x^2) \overset{AM-GM}{\le} \left(\frac{2x^2 + 1 - x^2 + 1 - x^2}{3}\right)^3$$

$$2x^2(1-x^2)^2 \le \frac{8}{27} \Rightarrow x^2(1-x^2)^2 \le \frac{4}{3\sqrt{3}}$$

$$x(1-x^2) \le \frac{2}{3\sqrt{3}} \Rightarrow \frac{1}{x(1-x^2)} \ge \frac{3\sqrt{3}}{2},\ \frac{yzt}{x(1-x^2)} \ge \frac{\sqrt{3}}{2} yzt \quad (1)$$

Analogous:

$$\frac{ztx}{y(1-y^2)} \ge \frac{3\sqrt{3}}{2} ztx \quad (2)$$

$$\frac{txy}{z(1-z^2)} \ge \frac{3\sqrt{3}}{2} txy \quad (3)$$

$$\frac{xyz}{t(1-t^2)} \ge \frac{3\sqrt{3}}{2} xyz \quad (4)$$

By adding (1); (2); (3); (4):

$$\frac{yzt}{x(1-x^2)} + \frac{ztx}{y(1-y^2)} + \frac{txy}{z(1-z^2)} + \frac{xyz}{t(1-t^2)} \ge$$

$$\ge \frac{3\sqrt{3}}{2}(yzt + ztx + txy + xyz) = \frac{3\sqrt{3}}{2} \cdot \frac{4}{3\sqrt{3}} = 2$$

Equality holds for $x = y = z = t = \frac{1}{\sqrt{3}}$.

6.64 If $a, b, c, d > 0, a + b + c + d = 1$ then:

$$\sqrt[4]{\left(\frac{1}{a^2} - a^2\right)\left(\frac{1}{b^2} - b^2\right)\left(\frac{1}{c^2} - c^2\right)\left(\frac{1}{d^2} - d^2\right)} \geq \frac{255}{16}$$

Daniel Sitaru

Solution (Soumava Chakraborty)

$$\prod\left(\frac{1}{a^2} - a^2\right) = \prod\left(\frac{1 - a^4}{a^2}\right) = \prod\left(\frac{(1-a)(1+a)(1+a^2)}{a^2}\right)$$

$$= \prod\left(\frac{(b+c+d)(2a+b+c+d)(1+a^2)}{a^2}\right) (\because 1 = a+b+c+d)$$

$$\Rightarrow LHS \overset{(1)}{=} \sqrt[4]{\frac{\{\prod(b+c+d)(2a+b+c+d)\}\{\prod(1+a^2)\}}{(abcd)^2}}$$

Now, $\prod(b+c+d) = (b+c+d)(c+d+a)(d+a+b)(a+b+c)$

$$\overset{A-G}{\geq} \left(3\sqrt[3]{bcd}\right)\left(3\sqrt[3]{cda}\right)\left(3\sqrt[3]{dab}\right)\left(3\sqrt[3]{abc}\right) = 3^4(abcd)$$

$\Rightarrow \prod(b+c+d) \overset{(i)}{\geq} 3^4(abcd)$. *Again,* $\prod(2a+b+c+d) =$

$$= (2a+b+c+d)(2b+c+d+a)(2c+d+a+b)(2d+a+b+c)$$

$$\overset{A-G}{\geq} \left(5\sqrt[5]{a^2bcd}\right)\left(5\sqrt[5]{b^2cda}\right)\left(5\sqrt[5]{c^2dab}\right)\left(5\sqrt[5]{d^2abc}\right) = 5abcd$$

$$\Rightarrow \prod(2a+b+c+d) \overset{(ii)}{\geq} 5^4(abcd)$$

(i)·(ii)$\Rightarrow \prod\{(b+c+d)(2a+b+c+d)\} \overset{(iii)}{\geq} 15^4(abcd)^2$

(iii), (1)$\Rightarrow$*LHS*$\geq \sqrt[4]{\frac{15^4(abcd)^2\{\prod(1+a^2)\}}{(abcd)^2}} = 15\sqrt[4]{\prod(1+a^2)} \overset{?}{\geq} \frac{255}{16}$

$$\Rightarrow \frac{1}{4}\ln\left\{\prod(1+a^2)\right\} \overset{?}{\geq} \ln\frac{17}{16} \Leftrightarrow \sum \ln(1+a^2) \underset{(2)}{\overset{?}{\geq}} 4\ln\frac{17}{16}$$

Obviously, $\because a,b,c,d > 0 | \sum a = 1 \therefore 0 < a,b,c,d < 1$

Let $f(x) = \ln(1+x^2)\ \forall x \in (0,1)$. *Then,* $f''(x) = \frac{2(1-x^2)}{(1+x^2)^2} > 0$

$\Rightarrow f(x)$ *is convex* $\therefore \sum \ln(1+a^2) \overset{Jensen}{\geq} 4\ln\left(1+\left(\frac{\sum a}{4}\right)^2\right) = 4\ln\left(1+\frac{1}{16}\right)$

$= 4\ln\frac{17}{16} \Rightarrow$ *(2) is true* $\Rightarrow$ *given inequality is true (Proved)*

6.65 If $0 < a,b,c \leq \frac{\pi}{2}$ then:

$$(1+cos^2a)(1+cos^2b)(1+cos^2c)(sina)^{2sin^2a}(sinb)^{2sin^2b}(sinc)^{2sin^2c} \geq 1$$

Daniel Sitaru

Solution (Adrian Popa)

Let: $f(x) = (1+x)(1-x)^{1-x}; x \in (0,1]$

$$\Rightarrow logf(x) = log(1+x) + (1-x)log(1-x) = g(x)$$
$$g'(x) = \frac{1}{x+1} - log(1-x) - 1$$

g(x)=$\frac{x^2+3x}{(x+1)^2(1-x)}$; $g"(x) = 0 \Leftrightarrow x = 0, (x \neq -3)$

So, $g(x) > 0, \forall x > 0 \Rightarrow logf(x) > 0 \Rightarrow f(x) > 1$

$$\Rightarrow (1+x)(1-x)^{1-x} > 1; \forall x \in (0,1]$$

If $x = cos^2a \Rightarrow (1+cos^2a)(1-cos^2a)^{1-cos^2a} > 1$

$$\Rightarrow (1+cos^2a)(sin^2a)^{sin^2a} > 1 \Rightarrow (1+cos^2a)(sina)^{2sin^2a} > 1 \text{ and}$$

analogs. Then:

$$(1+cos^2a)(1+cos^2b)(1+cos^2c)(sina)^{2sin^2a}(sinb)^{2sin^2b}(sinc)^{2sin^2c} \geq 1$$

6.66 If $x_i > 0, i \in \overline{1,2020}, x_1 + x_2 + \dots + x_{2020} = \frac{1}{x_1} + \frac{1}{x_2} + \dots + \frac{1}{x_{2020}}$

then:

$$(x_1 + x_3 + x_5 + \dots + x_{2019})(x_2 + x_4 + x_6 + \dots + x_{2020}) \geq 1010^2$$

Rahim Shahbazov

Solution

$$\text{Let: } \begin{cases} A = x_1 + x_3 + x_5 + \dots + x_{2019} \\ B = x_2 + x_4 + x_6 + \dots + x_{2020} \end{cases}$$

We must show that: $A \cdot B \geq 1010^2$

$$A + B = \left(\frac{1}{x_1} + \frac{1}{x_3} + \dots + \frac{1}{x_{2019}}\right) + \left(\frac{1}{x_2} + \frac{1}{x_4} + \dots + \frac{1}{x_{2020}}\right) \overset{Bergstrom}{\geq}$$

$$\geq \frac{1010^2}{A} + \frac{1010^2}{B} = 1010^2 \cdot \frac{A+B}{A \cdot B} \Rightarrow A \cdot B \geq 1010^2$$

6.67 If $n \in \mathbb{N}, n \geq 1$ then:

$$\frac{1}{n!} \leq log^n\left(1 + \prod_{k=1}^{n}\left(e^{\frac{1}{k}} - 1\right)^{\frac{1}{n}}\right) \leq \left(\frac{H_n}{n}\right)^n$$

Daniel Sitaru

Solution (Florică Anastase)

$$\frac{x}{1+x} \leq log(1+x) \leq x, \forall x > -1; \quad (1)$$

$$\prod_{k=1}^{n}\left(e^{\frac{1}{k}} - 1\right) \geq \prod_{k=1}^{n}\frac{1}{k} \to \prod_{k=1}^{n}\left(e^{\frac{1}{k}} - 1\right)^{\frac{1}{n}} \geq \frac{1}{\sqrt[n]{n!}}; (2)$$

$$\left(1+\prod_{k=1}^{n}\left(e^{\frac{1}{k}}-1\right)^{\frac{1}{n}}\right)^{n} \overset{Huygens}{\leq} \prod_{k=1}^{n}\left(1+e^{\frac{1}{k}}-1\right)=\prod_{k=1}^{n} e^{\frac{1}{k}}=e^{H_n} \rightarrow$$

$$log\left(1+\prod_{k=1}^{n}\left(e^{\frac{1}{k}}-1\right)^{\frac{1}{n}}\right) \leq \frac{H_n}{n};(3)$$

From $(1),(2),(3)$ *we get:*

$$\frac{1}{n!} \leq log^n\left(1+\prod_{k=1}^{n}\left(e^{\frac{1}{k}}-1\right)^{\frac{1}{n}}\right) \leq \left(\frac{H_n}{n}\right)^n$$

6.68 If $0<y<x<2y$ then:

$$x+y<3y^2\sqrt{\frac{2}{4y^2-x^2}}$$

$$x(x+y)>3(x-y)\sqrt{4y^2-x^2}$$

Daniel Sitaru

Solution(Daoudi Abdessattar)

$$0<y<x<2y$$

$$x+y<3y^2\sqrt{\frac{2}{4y^2-x^2}} \quad (1)$$

Using AM-GM:

$$\sqrt{(4y^2-x^2)\frac{(x+y)^2}{2}} \overset{AM-GM}{\leq} \frac{9}{4}y^2+\frac{1}{2}xy-\frac{1}{4}x^2$$

$$\frac{9}{4}y^2+\frac{1}{2}xy-\frac{1}{4}x^2<3y^2 \Leftrightarrow 3y^2-2xy+x^2=2y^2+(x-y)^2>0 \text{ true}$$

$$x(x+y)>3(x-y)\sqrt{4y^2-x^2} \quad (2)$$

$$Rhs \overset{AM-GM}{<} 6y(x-y)$$

$$6y(x-y) < x(x+y) \Leftrightarrow 6y^2 - 5xy + x^2 > 0 \Leftrightarrow \left(\frac{x}{y} - 2\right)\left(\frac{x}{y} - 3\right) > 0 \textit{ true}$$

$$\textit{since } \frac{x}{y} < 2$$

6.69 If $a, b, c > 0$ then:

$$\frac{(a^2+b^2+c^2)^9}{2(ab+bc+ca)(a+b+c)^7} \le \frac{a^{10}}{b+c} + \frac{b^{10}}{c+a} + \frac{c^{10}}{a+b}$$

Daniel Sitaru

Solution (Tran Hong)

Using Hölder's inequality:

$$(u^3+v^3+w^3)(x^3+y^3+z^3)(m^3+n^3+p^3) \ge (uxm+vyn+wzp)^3 \quad (*)$$

(with: $u, v, w, x, y, z, m, n, p > 0$)

Now, choose:

$$u^3 = x^3 = a; v^3 = y^3 = b, w^3 = z^3 = c; m^3 = a^4; n^3 = b^4; p^3 = c^4$$

Then: (*)$\leftrightarrow (a+b+c)^2(a^4+b^4+c^4) \ge (a^2+b^2+c^2)^3$ *(1)*

Lastly, choose: $u^3 = \frac{a^{10}}{b+c}; v^3 = \frac{b^{10}}{a+c}; w^3 = \frac{c^{10}}{a+b}; x^3 = a(b+c); y^3 = (b+c);$

$$z^3 = c(a+b); m^3 = a; n^3 = b; p^3 = c$$

Then: (*)$\leftrightarrow \left(\sum \frac{a^{10}}{b+c}\right)\left(\sum a(b+c)\right)\left(\sum a\right) \ge (a^4+b^3+c^4)^3 \overset{(1)}{\ge} \frac{(a^2+b^2+c^2)^9}{(a+b+c)^6}$

$\leftrightarrow \left(\sum \frac{a^{10}}{b+c}\right)\left(2\sum a\right)\left(\sum a\right)^7 \ge (a^2+b^2+c^2)^9$. *Proved. Equality* $\leftrightarrow a = b = c$.

6.70 If $a, b, c > 0, \frac{1}{a} + \frac{1}{b} + \frac{1}{c} = 6$ then:

$$\frac{3a}{4a^2+2a+1} + \frac{3b}{4b^2+2b+1} + \frac{3c}{4c^2+2c+1} \le a+b+c$$

Daniel Sitaru

Solution(Şerban George Florin)

$$\frac{a+b+c}{3} \geq \frac{3}{\frac{1}{a}+\frac{1}{b}+\frac{1}{c}} \Rightarrow a+b+c \geq \frac{9}{\frac{1}{a}+\frac{1}{b}+\frac{1}{c}} = \frac{9}{6} = \frac{3}{2}$$

(Ma≥Mb)$\Rightarrow a+b+c \geq \frac{3}{2}$

$$4a^2+2a+1>0, \Delta = 4-16=-12<0; \forall a>0$$

$$\frac{3a}{4a^2+2a+1} \leq \frac{1}{2} \Rightarrow 4a^2+2a+1 \geq 6a \Rightarrow 4a^2-4a+1 \geq 0$$

$\Rightarrow (2a-1)^2 \geq 0, (\forall)a>0$, *true.*

$\Rightarrow \sum_{a,b,c} \frac{3a}{4a^2+2a+1} \leq \sum_{a,b,c} \frac{1}{2} = \frac{3}{2} \leq a+b+c$ *true.*

6.71 If $x, y \in \left(0, \frac{\pi}{2}\right)$ then:

$$\frac{3}{\sin x} - \frac{1}{2\sin y \cos x} + \frac{1}{2\cos x \cos y} < \frac{6}{\sin 2x \sin 2y \cos x}$$

Daniel Sitaru

Solution(Tran Hong)

$$\frac{3}{\sin x} - \frac{1}{2\sin y \cos x} + \frac{1}{2\cos x \cos y} < \frac{6}{\sin 2x \sin 2y \cos x}$$

$$\Leftrightarrow \frac{6\cos^2 x \sin 2y + \sin 2x(\sin y - \cos y)}{\sin 2x \sin 2y \cos x} < \frac{6}{\sin 2x \sin 2y \cos x}$$

Because: $0 < x, y < \frac{\pi}{2} \Rightarrow \sin 2x, \sin 2y, \cos x > 0$. *We need to prove:*

$$6\cos^2 x \sin 2y + \sin 2x(\sin y - \cos y) < 6$$

$$\Leftrightarrow 3(1+\cos 2x)\sin 2y + \sin 2x(\sin y - \cos y) < 6$$

$$\Leftrightarrow 3\cos 2x \sin 2y + \sin 2x(\sin y - \cos y) < 3 \quad (*)$$

$$LHS_{(*)} \leq 3\left|\cos 2x|\sin 2y| + |\sin 2x||\sin y - \cos y|\right| \overset{(BCS)}{\leq} \sqrt{9\sin^2 2y + 1 - \sin 2y} \overset{(1)}{\leq} 3$$

(1)$\Leftrightarrow 9\sin^2 2y - \sin 2y < 8 \quad (t = \sin 2y, 0 < t < 1)$

$$\Leftrightarrow 9t^2 - t - 8 < 0 \Leftrightarrow 9(t-1)\left(t+\frac{8}{9}\right) < 0;$$

(True) $\Rightarrow$ *(1) true* $\Rightarrow$ *(*) true.*

6.72 If $a,b\in[0,1]$; $a\le b$ then:

$$4\sqrt{ab}\le a\left(\left(\frac{b}{a}\right)^{\sqrt{ab}}+\sqrt{\left(\frac{b}{a}\right)^{a+b}}\right)+b\left(\left(\frac{a}{b}\right)^{\sqrt{ab}}+\sqrt{\left(\frac{a}{b}\right)^{a+b}}\right)\le 2(a+b)$$

Daniel Sitaru

Solution

Let be $f:[0,1]\to\mathbb{R};\ f(x)=a\left(\frac{b}{a}\right)^x+b\left(\frac{a}{b}\right)^x$

$$f'(x)=a\left(\frac{b}{a}\right)^x\ln\frac{b}{a}+b\left(\frac{a}{b}\right)^x\ln\frac{a}{b}=\ln\frac{b}{a}\left[a\left(\frac{b}{a}\right)^x-b\left(\frac{a}{b}\right)^x\right]$$

$$f'(x)=0\Rightarrow a\left(\frac{b}{a}\right)^x=b\left(\frac{a}{b}\right)^x\Rightarrow a\left(\frac{b}{a}\right)^{2x}=b$$

$$\left(\frac{b}{a}\right)^{2x}=\left(\frac{b}{a}\right)^1\Rightarrow 2x=1\Rightarrow x=\frac{1}{2}$$

x	0	$\frac{1}{2}$	1
$f'(x)$	$-\ -\ -\ -\ -\ -\ -\ -\ -\ -$	0	$+\ +\ +\ +\ +\ +\ +\ +$
$f(x)$	$a+b$ ↘	$2\sqrt{ab}$ ↗	$a+b$

$$\Rightarrow 2\sqrt{ab}\le a\left(\frac{b}{a}\right)^x+b\left(\frac{a}{b}\right)^x\le a+b\quad (1)$$

For $x=\sqrt{ab}\in[a,b]\subseteq[0,1]$ *in (1):*

$$2\sqrt{ab}\le a\left(\frac{b}{a}\right)^{\sqrt{ab}}+b\left(\frac{a}{b}\right)^{\sqrt{ab}}\le a+b\quad (2)$$

For $x=\frac{a+b}{2}\in[a,b]\subseteq[0,1]$ *in (1):*

$$2\sqrt{ab}\le a\left(\frac{b}{a}\right)^{\frac{a+b}{2}}+b\left(\frac{a}{b}\right)^{\frac{a+b}{2}}\le a+b\quad (3)$$

By adding (2); (3):

$$4\sqrt{ab} \le a\left(\left(\frac{b}{a}\right)^{\sqrt{ab}} + \left(\frac{b}{a}\right)^{\frac{a+b}{2}}\right) + b\left(\left(\frac{a}{b}\right)^{\sqrt{ab}} + \left(\frac{a}{b}\right)^{\frac{a+b}{2}}\right) \le 2(a+b)$$

$$4\sqrt{ab} \le a\left(\left(\frac{b}{a}\right)^{\sqrt{ab}} + \sqrt{\left(\frac{b}{a}\right)^{a+b}}\right) + b\left(\left(\frac{a}{b}\right)^{\sqrt{ab}} + \sqrt{\left(\frac{a}{b}\right)^{a+b}}\right) \le 2(a+b)$$

6.73 If $x \in \mathbb{R}$ then:

$$\sqrt{\mathbf{4x^2+3}} + \sqrt{\mathbf{x^2+x+1}} + \sqrt{\mathbf{x^2-x+1}} < 3\sqrt{(\mathbf{4x^2+3})(\mathbf{x^4+x^2+1})}$$

Daniel Sitaru

Solution (Ali Jaffal)

Let $a = \sqrt{4x^2+3}$; $b = \sqrt{x^2+x+1}$ *and* $c = \sqrt{x^2-x+1}$

$$a+b+c-3abc = a-abc+b-abc+c-abc =$$

$$= a(1-bc)+b(1-ac)+c(1-ab)$$

We have $bc = \sqrt{(x^2+x+1)(x^2-x+1)} = \sqrt{x^4+x^2+1} > 1$

So, $1-bc<0$

$$ac = \sqrt{(4x^2+3)(x^2-x+1)} > \sqrt{(4x^2+3)\left(\left(x-\frac{1}{2}\right)^2+\frac{3}{4}\right)} > \sqrt{\frac{9}{4}} = \frac{3}{2} > 1$$

So, $1-ac<0$

$ab = \sqrt{(4x^2+3)(x^2+x+1)} > \sqrt{3} > 1$ *then* $1-ab<0$

So, $a+b+c-3abc<0$; $a+b+c<3abc$ *therefore* $\sqrt{4x^2+3}+$

$$\sqrt{x^2+x+1}+$$

$$+\sqrt{x^2-x+1} < 3\sqrt{(4x^2+3)(x^4+x^2+1)}$$

6.74 If $x, y, z > 0$ then:

$$\frac{\sqrt[3]{\mathbf{xz^2}}}{\mathbf{y}} + \frac{\sqrt[3]{\mathbf{yx^2}}}{\mathbf{z}} + \frac{\sqrt[3]{\mathbf{zy^2}}}{\mathbf{x}} \le \frac{\mathbf{x}}{\mathbf{y}} + \frac{\mathbf{y}}{\mathbf{z}} + \frac{\mathbf{z}}{\mathbf{x}}$$

Daniel Sitaru

Solution (Tran Hong)

$$\sqrt[3]{xz^2} \overset{AM-GM}{\leq} \frac{x+z+z}{3} = \frac{x}{3} + \frac{2z}{3}, \qquad \sqrt[3]{yx^2} \overset{AM-GM}{\leq} \frac{y}{3} + \frac{2x}{3}$$

$$\sqrt[3]{zy^2} \overset{AM-GM}{\leq} \frac{z}{3} + \frac{2y}{3}$$

$$\rightarrow LHS \leq \frac{1}{y}\left(\frac{x}{3} + \frac{2z}{x}\right) + \frac{1}{z}\left(\frac{y}{3} + \frac{2x}{3}\right) + \frac{1}{x}\left(\frac{z}{x} + \frac{2y}{3}\right)$$

$$= \frac{x}{3y} + \frac{y}{3z} + \frac{z}{3x} + \frac{2y}{3x} + \frac{2z}{3y} + \frac{2x}{3z}$$

Suppose: $0 < x \leq y \leq z$. *We must show that:* $\frac{x}{3y} + \frac{y}{3z} + \frac{z}{3x} + \frac{2y}{3x} + \frac{2z}{3y} + \frac{2x}{3z} \leq$

$$\frac{x}{y} + \frac{y}{z} + \frac{z}{x}$$

$$\leftrightarrow \frac{2y}{3x} + \frac{2z}{3y} + \frac{2x}{3z} \leq \frac{2x}{3y} + \frac{2y}{3z} + \frac{2z}{3x} \leftrightarrow zy^2 + xz^2 + yx^2 \leq yz^2 + zx^2 + xy^2$$

$\leftrightarrow (z-y)(y-x)(z-x) \geq 0$. *It is true because:* $0 < x \leq y \leq z$

Proved. Equality $\leftrightarrow x = y = z$.

6.75

$$\boldsymbol{A = a+b+c, B = \sqrt[3]{(a+b)(b+c)(c+a)}, C = \sqrt[3]{abc}, a,b,c} > 0$$

If $0 < x \leq y \leq z$ **then:**

$$\frac{\boldsymbol{6Ax + 9By + 18Cz}}{\boldsymbol{2A + 3B + 6C}} \leq \boldsymbol{x + y + z}$$

When does the equality holds?

Daniel Sitaru

Solution(Marian Ursărescu)

$2A \geq 3B \Leftrightarrow 2(a+b+c) \geq 3\sqrt[3]{(a+b)(b+c)(c+a)}$, *true because*

$$\sqrt[3]{(a+b)(b+c)(c+a)} \leq \frac{2(a+b+c)}{3}$$

$$3B \geq 6C \Leftrightarrow B \geq 2C \Leftrightarrow \sqrt[3]{(a+b)(b+c)(c+a)} \geq 2\sqrt[3]{abc}$$

$\Leftrightarrow (a+b)(b+c)(c+a) \geq 8abc$, *true because:*

$$a+b\geq 2\sqrt{ab}, b+c\geq 2\sqrt{bc}, c+a\geq 2\sqrt{ac}\Rightarrow$$

$$0<x\leq y\leq z \text{ and } 2A\geq 3B\geq 6C \text{ from Cebyshev}\Rightarrow$$

$$(x+y+z)(2A+3B+6C)\geq 3(2Ax+3By+6Cz)\Leftrightarrow$$

$$(x+y+z)(2A+3B+6C)\geq 6Ax+9By+18Cz$$

$$\text{Equality for } a=b=c\Rightarrow A=3a, B=2a, C=a$$

$$(x+y+z)\cdot 18a=18a(x+y+z)\Leftrightarrow x+y+z=x+y+z\Rightarrow 0<x\leq y\leq z$$

6.76 If $a,b,c,d>1, abcd=8$ then:

$$\log_b(ab)\cdot\log_c(bc)\cdot\log_d(cd)\cdot\log_a(da)\geq 3(\log_2 2+\log_b 2+\log_c 2+\log_d 2)$$

Daniel Sitaru

Solution(Adrian Popa)

$$a,b,c,d>1; abcd=8$$

$$\underbrace{\log_b(ab)\cdot\log_c(bc)\cdot\log_d(cd)\cdot\log_a(da)}_{A}\geq\underbrace{3(\log_a 2+\log_b 2+\log_c 2+\log_d 2)}_{B}$$

$$B=\log_a 8+\log_b 8+\log_c 8+\log_d 8$$

$$=\log_a abcd+\log_a abcd+\log_c abcd+\log_d abcd$$

$$=1+\log_a b+\log_a c+\log_a d+\frac{1+\log_a b+\log_a c+\log_a d}{\log_a b}+$$

$$+\frac{1+\log_a b+\log_a c+\log_a d}{\log_a c}+\frac{1+\log_a b+\log_a c+\log_a d}{\log_a d}$$

$$\text{We denote } \log_a b=m; \log_a c=n; \log_a d=p$$

$$B=1+m+n+p+\frac{1}{m}+1+\frac{n}{m}+\frac{p}{m}+\frac{1}{n}+\frac{m}{n}+1+\frac{p}{n}+\frac{1}{p}+\frac{m}{p}+\frac{n}{p}+1$$

$$A=\frac{1+\log_a b}{\log_a b}\cdot\frac{\log_a b+\log_a c}{\log_a c}\cdot\frac{\log_a c+\log_a d}{\log_a d}\cdot(\log_a d+1)\Rightarrow$$

$$\Rightarrow A=\left(\frac{1}{m}+1\right)\left(\frac{m}{n}+1\right)\left(\frac{n}{p}+1\right)(p+1)=\left(\frac{1}{n}+\frac{1}{m}+\frac{m}{n}+1\right)\left(n+\frac{n}{p}+p+1\right)=$$

$$=1+\frac{1}{p}+\frac{p}{n}+\frac{1}{n}+\frac{n}{m}+\frac{n}{mp}+\frac{p}{m}+\frac{1}{m}+m+\frac{m}{p}+\frac{mp}{n}+\frac{m}{n}+n+\frac{n}{p}+p+1$$

Being long expressions, in order to be able to compare A and B we will write them one under the other:

$$A=1+\frac{1}{p}+\frac{p}{n}+\frac{1}{n}+\frac{n}{m}+\frac{n}{mp}+\frac{p}{m}+\frac{1}{m}+m+\frac{m}{p}+\frac{mp}{n}+\frac{m}{n}+n+\frac{n}{p}+p+1$$

$$B=1+\frac{1}{p}+\frac{p}{n}+\frac{1}{n}+\frac{n}{m}\quad\frac{p}{m}+\frac{1}{m}+m+\frac{m}{p}\quad\frac{m}{n}+n+\frac{n}{p}+p+1+1+1$$

So, we have to prove that: $\frac{n}{mp}+\frac{mp}{n}\ge 2\Rightarrow n^2+(mp)^2>2mnp$

$$n^2-2mnp+(mp)^2\ge 0,(n-mp)^2\ge 0\quad (A)$$

6.77 If $a,b,c>0,a+b+c=3$ then:

$$a(a+c)e^{\frac{1}{a}}+b(b+a)e^{\frac{1}{b}}+c(c+b)e^{\frac{1}{c}}\ge 6e$$

Daniel Sitaru

Solution (Michael Sterghiou)

Generalization of Dan Sitaru's problem, RMM, April 2019

If $a,b,c>0$ *then* $\sum_{cyc}a(a+c)\cdot e^{\frac{1}{a}}\ge 2\cdot\left(\sum_{cyc}a\right)\cdot e$ *(1)*

Let $(p,q,r)=\left(\sum_{cyc}a,\sum_{cyc}ab,abc\right)$. *The function* $f(t)=e^t$ *is convex on* $(0,\infty)$ *therefore by applying the generalized Jensen on (1) we have:*

$$LHS\ (1)\ge\left[\sum_{cyc}a(a+c)\right]\cdot e^{\frac{\sum_{cyc}a(a+c)\frac{1}{a}}{\sum_{cyc}a(a+c)}}\overset{?}{\ge} RHS\ of\ (1)$$

where the "weights" are $a(a+c),b(b+a),c(c+b)$. *The last inequality reduces to:*

$$(p^2-2q+q)\cdot e^{\frac{2p}{p^2-2q+q}}\ge 2p\cdot e$$

$$[as\ \sum_{cyc}a^2=p^2-2q]\ or\ \frac{p^2-q}{2p}\ge e^{1-\frac{2p}{p^2-2q}}\quad (3)$$

$$Let\ \frac{p^2-q}{2p}=6$$

Case 1: $q>1$: *then* $1-\frac{1}{q}<1$. *From the unknown inequality:*

$$e^{\theta} < \frac{1}{1-\theta}, \theta < 1 \text{ we have: } e^{1-\frac{1}{q}} < \frac{1}{1-\left(1-\frac{1}{q}\right)} = q \text{ so (3) holds:}$$

Case 2: $q = 1$*: Then (3) holds as equality*

$$\text{Case 3: } q < 1. \text{ Then (3)} \to q > e^{1-\frac{1}{q}} = \frac{1}{e^{\frac{1}{q}-1}} \left(\frac{1}{q} > 1\right) \text{ or}$$

$$e^{\frac{1}{q}-1} \geq \frac{1}{q}. \text{ Let } \frac{1}{q} = \theta > 1 \text{ (3)} \to e^{\theta-1} > \theta \text{ which holds for } \theta > 1.$$

We are done. Equality holds for $a = b = c = 1$.

6.78 If $m, n, p \in \mathbb{N}$ then:

$$3\sqrt{3}\left(\frac{m^3}{(m+3)!} + \frac{n^5}{(n+5)!} + \frac{p^7}{(p+7)!}\right) < \sqrt{(m!)^2 + (n!)^2 + (p!)^2}$$

Daniel Sitaru

Solution(Tran Hong)

For $m \in \mathbb{N} \Rightarrow$

$$3\sqrt{3}\left(\frac{m^3}{(m+3)!}\right) \overset{(*)}{\leq} \frac{\sqrt{3}}{3} \cdot m! \Leftrightarrow 9 \cdot \frac{m^3}{(m+3)!} < m! \Leftrightarrow 9m^3 < m!\,(m+3)!$$

Which is true because:

If $m = 0$ then: $0!\,(0+3)! = 1 > 0 = 0 \cdot 9^3$

If $m \geq 1, m \in \mathbb{N}$ *then:* $m! \geq m$ *and*

$$(m+3)! = (m+3)(m+2)(m+1)m! \geq 4 \cdot 3(m+1)m! \geq 12(m+1)m! \geq 9(m+1)m > 9m^2 \Rightarrow m!\,(m+3)! > m \cdot 9m^2 = 9m^3 \Rightarrow (*) \text{ true.}$$

Similary:

$$3\sqrt{3}\left(\frac{n^5}{(n+5)!}\right) < \frac{\sqrt{3}}{3!} \cdot n!, \forall n \in \mathbb{N} \text{ and } 3\sqrt{3}\left(\frac{p^7}{(p+7)!}\right) < \frac{\sqrt{3}}{3!} \cdot p!, \forall p \in \mathbb{N}$$

So,

$$LHS < \frac{\sqrt{3}}{3}(m! + n! + p!) \overset{BCS}{\leq} \frac{\sqrt{3}}{3} \cdot \sqrt{3} \cdot \sqrt{(m!)^2 + (n!)^2 + (p!)^2} =$$

$$= \sqrt{(m!)^2 + (n!)^2 + (p!)^2}. \text{ Proved.}$$

6.79 If $m,n\in\mathbb{N}-\{0\}, gcd(m,n)=1, \varphi$ −Euler's totient function, $\tau(n)$ −number of positive divisors of n then:

$$\sqrt{\varphi(mn)\cdot\tau(m)\cdot\tau(n)}\ge\frac{2mn}{m+n}$$

Rajeev Rastogi

Solution (George Florin Şerban)

$$gcd(m,n)=1\to\varphi(m,n)=\varphi(m)\cdot\varphi(n)$$

$$m=p_1^{x_1}\cdot p_2^{x_2}\cdot\ldots\cdot p_k^{x_k};\ p_1,p_2,\ldots,p_k\ge 2\text{ −prime numbers and } x_1,x_2,\ldots,x_k\in\mathbb{N}^*$$

$$n=q_1^{y_1}\cdot q_2^{y_2}\cdot\ldots\cdot q_r^{y_r};\ q_1,q_2,\ldots,q_r\ge 2\text{ −prime numbers and}$$

$$y_1,y_2,\ldots,y_r\in\mathbb{N}^*$$

$$(x_i+1)\left(1-\frac{1}{p_i}\right)\ge(1+1)\left(1-\frac{1}{2}\right)=1$$

$$(y_i+1)\left(1-\frac{1}{q_i}\right)\ge(1+1)\left(1-\frac{1}{2}\right)=1$$

$$\sqrt{\varphi(mn)\cdot\tau(m)\cdot\tau(n)}=\sqrt{\varphi(m)\cdot\varphi(n)\cdot\tau(m)\cdot\tau(n)}=$$

$$=\sqrt{m\left(1-\frac{1}{p_1}\right)\cdot\ldots\cdot\left(1-\frac{1}{p_k}\right)n\left(1-\frac{1}{q_1}\right)\cdot\ldots\cdot\left(1-\frac{1}{q_r}\right)\prod_{i=1}^{k}(1+x_i)\prod_{j=1}^{r}(1+x_j)}=$$

$$=\sqrt{mn\left(\prod_{i=1}^{k}(1+x_i)\left(1-\frac{1}{p_i}\right)\right)\left(\prod_{j=1}^{r}(1+x_j)\left(1-\frac{1}{q_j}\right)\right)}\ge$$

$$\ge\sqrt{mn\cdot 1\cdot 1}=\sqrt{mn}\overset{Gm-Hm}{\ge}\frac{2mn}{m+n}$$

6.80 Prove without computer:

$$e^e(1-e^{\tan e})>e^\pi-\pi^\pi$$

Rovsen Pirguliyev

Solution (Abdallah Almalih)

Put $f(x)=(1+\tan^2 x)e^{\tan x+e}+\pi x^{\pi-1}$ *where* $x\in[e,\pi]$. *Clearly, we have* $f(x)>0$. *So,*

$$\int_e^{\pi} f(x)\,dx>0$$

But

$$\int_e^{\pi}(1+\tan^2 x)\,e^{\tan x+e}+\pi x^{\pi-1}dx=\left[e^{\tan x+e}+x^{\pi}\right]_e^{\pi}$$

$$=e^{\tan\pi+e}+\pi^{\pi}-\left(e^{\tan e+e}+e^{\pi}\right)$$

$$=e^{e}\left[e^{\tan\pi}-e^{\tan e}\right]-\left(e^{\pi}-\pi^{\pi}\right)=e^{e}\left(1-e^{\tan e}\right)-\left(e^{\pi}-\pi^{\pi}\right)>0$$

Hence $e^{e}\left(1-e^{\tan e}\right)>e^{\pi}-\pi^{\pi}$

6.81 If $a,b,c\geq 0$ then:

$$3(\sinh a+\sinh b+\sinh c)\geq(a+b+c)\left(\sqrt[3]{\cosh a}+\sqrt[3]{\cosh b}+\sqrt[3]{\cosh c}\right)$$

Daniel Sitaru

Solution (Soumava Chakraborty)

$$\text{Let } f(x)=\cosh x\ \forall x\geq 0$$

$$f'(x)=\sinh x=\frac{e^x-e^{-x}}{2}=\frac{e^{2x}-1}{2e^x}$$

$$=\frac{(e^x+1)(e^x-1)}{2e^x}\geq 0\ (\because e^x\geq 1\ as\ x\geq 0)$$

$\therefore f(x)$ *is an increasing* f^n*, WLOG, we may assume* $a\geq b\geq c$

Then, as $\cosh x$ *is an increasing* f^n, $\forall x\geq 0$, $\therefore \cosh a\geq\cosh b\geq\cosh c$

$$\Rightarrow\sqrt[3]{\cosh a}\geq\sqrt[3]{\cosh b}\geq\sqrt[3]{\cosh c}$$

$$\therefore\sum a\sqrt[3]{\cosh a}\overset{Chebyshev}{\geq}\frac{1}{3}\left(\sum a\right)\left(\sum\sqrt[3]{\cosh a}\right)$$

$$\Rightarrow\left(\sum a\right)\left(\sum\sqrt[3]{\cosh a}\right)\overset{(1)}{\leq}3\sum\left(a\sqrt[3]{\cosh a}\right)$$

(1) ⇒ it suffices to show: $\sum \sinh a \geq \sum\left(a\sqrt[3]{\cosh a}\right)$ *(i)*

For 2 positive $m \,\&\, n$*, let*

$$A = A(m,n) = \frac{m+n}{2}, G = G(m,n) = \sqrt{mn} \;\; \& \; L = L(m,n) = \frac{m-n}{\ln m - \ln n}$$

We have, $\sqrt[3]{G^2 A} \overset{(a)}{<} L$ *(E.B. Leach & M.C. Scholander)*

Now, $A(e^x, e^{-x}) = \cosh x, G(e^x, e^{-x}) = 1, L(e^x, e^{-x}) = \frac{e^x - e^{-x}}{2x} = \frac{\sinh x}{x}$

$\therefore$ *applying (a), we get,* $\sqrt[3]{\cosh x} < \frac{\sinh x}{x}, \forall x > 0$

$\therefore a, b, c > 0, \sinh a > a\sqrt[3]{\cosh a}$ *etc*$\Rightarrow \sum \sin h > \sum a\sqrt[3]{\cosh a}$ *(2)*

For $a = 0, \sinh a = 0 \,\&\, a\sqrt[3]{\cosh a} = 0 \Rightarrow \sinh a = a\sqrt[3]{\cosh a}$

Similarly, for $b \,\&\, c = 0, \sinh b = b\sqrt[3]{\cosh b} \;\; \& \sinh c = c\sqrt[3]{\cosh c}$

$\therefore$ *when* $a = b = c = 0, \sum \sinh a = \sum a\sqrt[3]{\cosh a} \;\; (= 0)$ *(3)*

Combining (2) & (3), (i) is true (Proved)

6.82 If $0 < a, b \leq \frac{\pi}{2}$ **then:**

$$\frac{2}{\pi} \leq \frac{2sin\left(\sqrt{ab}\right)}{(a+b)sin\left(\frac{\pi\sqrt{ab}}{a+b}\right)} \leq \frac{2sin\left(\frac{a+b}{2}\right)}{a+b}$$

Daniel Sitaru

Solution (Remus Florin Stanca)

$$\frac{2sin\left(\sqrt{ab}\right)}{(a+b)sin\left(\frac{\pi\sqrt{ab}}{a+b}\right)} = \frac{\frac{sin\left(\sqrt{ab}\right)}{\sqrt{ab}}}{\frac{sin\left(\frac{\pi\sqrt{ab}}{a+b}\right)}{\frac{\pi\sqrt{ab}}{a+b}}} \cdot \frac{2}{\pi}$$

Let: $f:\left(0;\frac{\pi}{2}\right] \to \mathbb{R}, f(x) = \frac{sinx}{x}; f'(x) = \frac{xcosx - sinx}{x^2};$

$\text{let } g(x) = xcosx - sinx \Rightarrow$

$$g'(x)=-x\sin x+\cos x-\cos x\le 0, g(0)=0\Rightarrow g(x)\le 0\Rightarrow f'(x)\le 0\Rightarrow$$

f *−decreasing.*

$$\frac{\pi}{a+b}\ge 1\Rightarrow\frac{\pi\sqrt{ab}}{a+b}\ge\sqrt{ab}\Rightarrow f\left(\frac{\pi\sqrt{ab}}{a+b}\right)\le f(\sqrt{ab})\Rightarrow\frac{\sin(\sqrt{ab})}{\sqrt{ab}}$$

$$\ge\frac{\sin\left(\frac{\pi\sqrt{ab}}{a+b}\right)}{\frac{\pi\sqrt{ab}}{a+b}}\Rightarrow\frac{\frac{\sin(\sqrt{ab})}{\sqrt{ab}}}{\frac{\sin\left(\frac{\pi\sqrt{ab}}{a+b}\right)}{\frac{\pi\sqrt{ab}}{a+b}}}\cdot\frac{2}{\pi}\ge\frac{2}{\pi};(1)$$

We must prove that:

$$\frac{\frac{\sin(\sqrt{ab})}{\sqrt{ab}}}{\frac{\sin\left(\frac{\pi\sqrt{ab}}{a+b}\right)}{\frac{\pi\sqrt{ab}}{a+b}}}\cdot\frac{2}{\pi}\le\frac{\sin\frac{a+b}{2}}{\frac{a+b}{2}}\Leftrightarrow\frac{\frac{\sin(\sqrt{ab})}{\sqrt{ab}}}{\frac{\sin\left(\frac{\pi\sqrt{ab}}{a+b}\right)}{\frac{\pi\sqrt{ab}}{a+b}}}\le\frac{\frac{\sin\frac{a+b}{2}}{\frac{a+b}{2}}}{\frac{\sin\frac{\pi}{2}}{\frac{\pi}{2}}}$$

$$\text{Let: } h(x)=\frac{\frac{\sin x}{x}}{\frac{\sin\left(\frac{\pi x}{a+b}\right)}{\frac{\pi x}{a+b}}}=\frac{\sin x}{x}\cdot\frac{\pi x}{a+b}\cdot\frac{1}{\sin\frac{\pi x}{a+b}}=\frac{\pi}{a+b}\cdot\frac{\sin x}{\sin\frac{\pi x}{a+b}};(2)$$

$$\text{Let: } f_1(x)=\frac{\sin x}{\sin\left(\frac{\pi x}{a+b}\right)}\Rightarrow f'_1(x)=\frac{\cos x\cdot\sin\frac{\pi x}{a+b}-\sin x\cdot\cos\frac{\pi x}{a+b}\cdot\frac{\pi x}{a+b}}{\sin^2\frac{\pi x}{a+b}}$$

$$\text{Let: } f_2(x)=\cos x\cdot\sin\frac{\pi x}{a+b}-\sin x\cdot\cos\frac{\pi x}{a+b}\cdot\frac{\pi x}{a+b}\Rightarrow$$

$$f'_2(x)=\sin x\cdot\sin\frac{\pi x}{a+b}\left(\frac{\pi^2}{a+b}-1\right)\ge 0\left(\text{we take the function for } x\le\frac{a+b}{2}\right)$$

6.83 If $a,b,c\in(0,1), 2(a+b+c)=3$ then:

$$\sum(3+(\log_a c)^4)\left(3+\frac{1}{(a+b)^4}\right)\ge 48$$

Daniel Sitaru

Solution (Șerban George Florin)

$$3+\frac{1}{(a+b)^4}=1+1+1+\frac{1}{(a+b)^4}\overset{(Ma\geq Mg)}{\geq}4\sqrt[4]{1\cdot 1\cdot 1\cdot\frac{1}{(a+b)^4}}=\frac{4}{a+b}$$

$$3+(\log_a c)^4=1+1+1+(\log_a c)^4\overset{(Ma\geq Mg)}{\geq}4\sqrt[4]{1\cdot 1\cdot 1(\log_a c)^4}=4\log_a c$$

$$\sum(3+(\log_a c)^4)\left(1+\frac{1}{(a+b)^4}\right)\geq\sum 16\cdot\frac{\log_a c}{a+b}=16\sum\frac{\log_a c}{a+b}$$

$$16\sum\frac{\log_a c}{a+b}\overset{(Ma\geq Mg)}{\geq}16\cdot 3\sqrt[3]{\frac{\prod\log_a c}{\prod(a+b)}}=$$

$$=\frac{48}{\sqrt[3]{(a+b)(b+c)(a+c)}}\overset{(Ma\geq Mg)}{\geq}\frac{48}{\frac{a+b+b+c+a+c}{3}}=$$

$$=\frac{48}{\frac{2(a+b+c)}{3}}=\frac{48}{\frac{3}{3}}=48$$

$$a,b,c\in(0,1)\Rightarrow\log_a c,\log_b c,\log_a b>0$$

6.84 If $x,y,z,t\geq 1$ then:

$$(\ln xy)(\ln^2 x+\ln^2 y-\ln x\ln y-\ln z\ln t)\geq(\ln zt)(\ln x\ln y+\ln z\ln t-\ln^2 z-\ln^2 t)$$

Daniel Sitaru

Solution (Soumava Chakraborty)

Let $a=\ln x, b=\ln y, c=\ln z, d=\ln t$

$(a,b,c,d\geq 0)$

Using this substitution, given inequality

$$\Leftrightarrow(a+b)(a^2+b^2-ab-cd)\geq(c+d)(ab+cd-c^2-d^2)$$

$$\Leftrightarrow a^3+b^3+c^3+d^3\overset{(1)}{\geq}abc+bcd+cda+dab$$

Now, $a^3+b^3+c^3=3abc+(a+b+c)(a^2+b^2+c^2-ab-bc-ca)\overset{(2)}{\geq}3abc,\forall a,b,c\geq 0$. *Similarly,* $b^3+c^3+d^3\overset{(3)}{\geq}3bcd,\forall b,c,d\geq 0, c^3+d^3+a^3\overset{(4)}{\geq}3cda,\forall c,d,a\geq 0\ \&$

$$d^3+a^3+b^3\overset{(5)}{\geq}3dab,\forall d,a,b\geq 0$$

(2)+(3)+(4)+(5) $\Rightarrow a^3+b^3+c^3+d^3\geq abc+bcd+cda+dab\Rightarrow$

(1) is true (Proved)

6.85 *If* $1\leq x<y$ *then:*

$$\frac{(y^5-x^5)(y^7-x^7)(y^9-x^9)}{(y^6-x^6)(y^8-x^8)(y^{10}-x^{10})}<\frac{21}{32}$$

Daniel Sitaru

Solution (Ravi Prakash)

Let $1\leq x<y,n,m\in\mathbb{N},n<m.$

By the Cauchy's mean value theorem:

$$\frac{y^n-x^n}{y^m-x^m}=\frac{n\alpha^{n-1}}{m\alpha^{m-1}}\text{ for some }\alpha\in(x,y)$$

$$=\frac{n}{m}\alpha^{n-m}=\frac{n}{m}\cdot\frac{1}{\alpha^{m-n}}$$

$$<\frac{n}{m}\ [\because\alpha>x\geq 1\Rightarrow\alpha>1]$$

$$\therefore\frac{y^5-x^5}{y^6-x^6}\cdot\frac{y^7-x^7}{y^8-x^8}\cdot\frac{y^9-x^9}{y^{10}-x^{10}}<\left(\frac{5}{6}\right)\left(\frac{7}{8}\right)\left(\frac{9}{10}\right)=\frac{21}{32}$$

Generalization(Sagar Kumar)

$$\Psi=\prod_{r=0}^{n}\left(\frac{y^{2r+1}-x^{2r+1}}{y^{2r+2}-x^{2r+2}}\right)<\frac{1}{4^{n+1}}\binom{n+1}{2n+2},1\leq x<y$$

$$\lim_{n\to\infty}(n+1)\Psi\leq\frac{1}{\sqrt{\pi}}$$

6.86 If $0\leq a,b,c\leq 1$ then:

$$\begin{vmatrix}1&1&1\\a&b&c\\e^{a^2}&e^{b^2}&e^{c^2}\end{vmatrix}<e-1$$

Daniel Sitaru

Solution (Soumava Chakraborty)

$$LHS = be^{c^2} - ce^{b^2} + ce^{a^2} - ae^{c^2} + ae^{b^2} - be^{a^2}$$

Case 1: Exactly one variable among $a, b, c = 0$. *WLOG we may assume* $a = 0$.

$$\therefore LHS = be^{c^2} - ce^{b^2} + c - b = b\left(e^{c^2} - 1\right) + c\left(1 - e^{b^2}\right)$$

$$\leq e^{c^2} - 1 + c\left(1 - e^{b^2}\right)\left(\because b \leq 1(\&b \geq 0)\&e^{c^2} - 1 \geq 0 \; as \; c \geq 0\right)$$

$$\overset{(1)}{\leq} e - 1 + c(1 - b^2) \; (\because c \leq 1 \; \& \; e^{c^2} \text{ is increasing on } [0,\infty))$$

Now, $e^{b^2} > 1 + b^2 \Rightarrow -e^{b^2} < -1 - b^2 \Rightarrow 1 - e^{b^2} < -b^2 \leq 0 \; \& \because c > 0$

$\therefore c\left(1 - e^{b^2}\right) < 0$ *(2); (1), (2)* $\Rightarrow LHS < e - 1$. *Analogous proof is evident if we assume* $b = 0$ *or* $c = 0$.

Case 2: Exactly 2 variables among $a, b, c = 0$. *WLOG we may assume* $a = b = 0$.

Then, LHS$= -c + c = 0 < e - 1$. *Analogous proof is evident if we assume* $b = c = 0$ *or* $c = a = 0$.

Case 3: $a, b, c > 0$

$$LHS = be^{c^2} - bc\left(\frac{e^{b^2}}{b}\right) + ce^{a^2} - ca\left(\frac{e^{c^2}}{c}\right) + ae^{b^2} - ab\left(\frac{e^{a^2}}{a}\right)$$

$$\leq (be + ce + ae) - \left(ab \cdot \frac{e^{a^2}}{a} + bc \cdot \frac{e^{b^2}}{b} + ca \cdot \frac{e^{c^2}}{c}\right)$$

$(\therefore c \leq 1$ *etc, &* e^{c^2} *etc is increasing on* $[0,\infty)$ *&* $b \geq 0$ *etc)*

$$\overset{(3)}{\leq} 3e - \left(ab \cdot \frac{e^{a^2}}{a} + bc \cdot \frac{e^{b^2}}{b} + ca \cdot \frac{e^{c^2}}{c}\right) \; (\because a, b, c \leq 1)$$

Let $f(x) = \frac{e^{x^2}}{x}, \forall x \in (0,1]; f'(x) = \frac{(2x^2-1)e^{x^2}}{x^2} \; \& \; f''(x) = \frac{(4x^4-2x^2+2)e^{x^2}}{x^3}$

$$f'(x) = 0 \; iff \; x = \frac{1}{\sqrt{2}} \; \& \; f''\left(\frac{1}{\sqrt{2}}\right) > 0$$

$\therefore f(x)$ *attains a minima at* $x = \frac{1}{\sqrt{2}} > 0 \; \& \because f(x)$ *never attains a maxima in* $(0,1]$,

$$\therefore f_{\min} = f\left(\frac{1}{\sqrt{2}}\right) \Rightarrow \frac{e^{x^2}}{x} \geq \frac{\sqrt{e}}{\frac{1}{\sqrt{2}}} = \sqrt{2e} \Rightarrow -\frac{e^{x^2}}{x} \leq -\sqrt{2e} \quad (4)$$

Now, $ab \leq 1 \Rightarrow ab - 1 \leq 0 \Rightarrow \left(-\frac{e^{a^2}}{a}\right)(ab-1) \leq 0 \Rightarrow -ab \cdot \frac{e^{a^2}}{a} \leq$

$$-\frac{e^{a^2}}{a} \underset{(a)}{\overset{by\,(4)}{\leq}} -\sqrt{2e}$$

Similarly, $-bc \cdot \frac{e^{b^2}}{b} \overset{(b)}{\leq} -\sqrt{2e}\ \&\ -ca \cdot \frac{e^{c^2}}{c} \overset{(c)}{\leq} -\sqrt{2e}$

(a)+(b)+(c) along with (3) $\Rightarrow LHS \leq 3e - 3\sqrt{2e} \overset{?}{<} e - 1$

$$\Leftrightarrow (2e+1)^2 \overset{?}{<} 18e \Leftrightarrow 4e^2 + 1 - 14e \overset{?}{<} 0 \quad (5)$$

Now, $e \leq \frac{11}{4} \Rightarrow 4e^2 < \frac{121}{4} \rightarrow$ *(i). Also,* $e > \frac{5}{2} \Rightarrow 14e > 35 \Rightarrow -14e < -35$ *(ii)*

(i)+(ii)$\Rightarrow 4e^2 + 1 - 14e < \frac{121}{4} + 1 - 35 = \frac{121+4-140}{4} = \frac{-15}{4} < 0 \Rightarrow$ *(5) is true*

$$\therefore LHS < e - 1$$

Case 4: $a = b = c = 0$

$LHS = 0 < e - 1 \therefore$ *combining all the cases,* $LHS < e - 1$ *(proved)*

6.87 If $0 \leq a, b, c, d \leq 2$ then:

$$\frac{9a}{1+bcd} + \frac{9b}{1+cda} + \frac{9c}{1+dab} + \frac{9d}{1+abc} + 9e^{abcd} \leq 8 + 9e^{16}$$

Daniel Sitaru

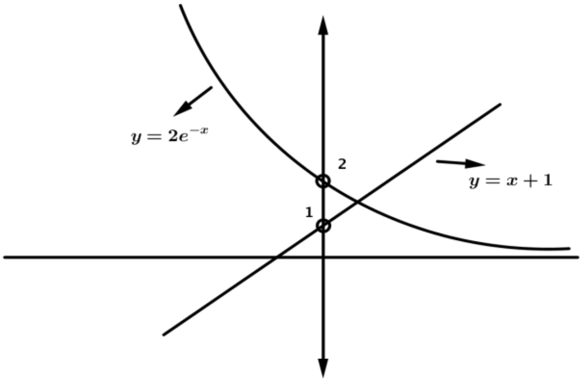
Solution (Soumava Chakraborty)

We shall show that: $\frac{9a}{1+bcd} + \frac{9}{4}e^{abcd} \overset{(1)}{\leq} 2 + \frac{9}{4}e^{16}$

LHS of (1) $\overset{(2)}{\leq} \frac{18}{1+bcd} + \frac{9}{4}e^{2bcd}\ (\because a \leq 2\ \&a \geq 0)$

Let $f(x) = \frac{18}{1+x} + \frac{9}{4}e^{2x}; f'(x) = \frac{9e^{2x}(x+1)^2 - 36}{2(x+1)^2}\ \&\ f''(x) = 9e^{2x} + \frac{36}{(x+1)^3}$

Now, $f'(x) = 0 \Rightarrow e^x(x+1) = 2 \Rightarrow x + 1 = 2e^{-x} \quad (3)$

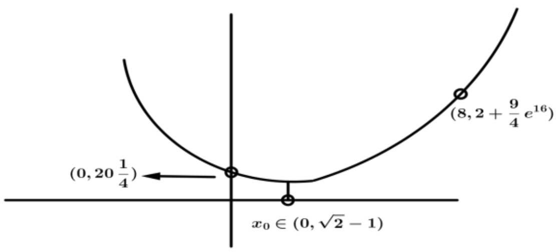


Also, $e^x = \frac{2}{x+1} \geq x + 1 \Rightarrow x \leq \sqrt{2} - 1 \therefore$ *(3) has only root & it* $\in (0, \sqrt{2} - 1) \Rightarrow$

$\Rightarrow f'(x) = 0$ *at one & only one value* $x_0 \in (0, \sqrt{2} - 1)$

$\& \because f''(x) > 0, \forall x \geq 0, \therefore f''(x_0) > 0 \Rightarrow f(x)$ *attains a minima at* $x_0 \in (0, \sqrt{2} - 1)$

Also, $f(0) = 18 + \frac{9}{4} = 20\frac{1}{4} \& f(8) = \frac{18}{1+8} + \frac{9}{4}e^{16} = 2 + \frac{9}{4}e^{16} > f(0) \& \because$ $f(x)$ *never attains a maxima in* $[0,8], \therefore$ *the graph of* $f(x)$ *in* $[0,8]$ *should be like below:*



Hence, it is clear that in $[0,8], f(x)_{\max} = f(8) = 2 + \frac{9}{4}e^{16} \Rightarrow \frac{18}{1+x} + \frac{9}{4}e^{2x} \leq 2 + \frac{9}{4}e^{16}$

$\Rightarrow \frac{18}{1+bcd} + \frac{9}{4}e^{2bcd} \leq 2 + \frac{9}{4}e^{16}$ *(putting* $x = bcd \& bcd \leq 8$*)*

$$\left.\begin{array}{l}\Rightarrow \dfrac{9a}{1+bcd}+\dfrac{9}{4}e^{abcd} \overset{by\ (2)}{\leq} 2+\dfrac{9}{4}e^{16} \\ Similarly, \dfrac{9b}{1+cda}+\dfrac{9}{4}e^{abcd} \leq 2+\dfrac{9}{4}e^{16} \\ \dfrac{9c}{1+dab}+\dfrac{9}{4}e^{abcd} \leq 2+\dfrac{9}{4}e^{16} \\ \dfrac{9d}{1+abc}+\dfrac{9}{4}e^{abcd} \leq 2+\dfrac{9}{4}e^{16}\end{array}\right\}$$

Adding the last 4, we obtain the desired inequality (proved)

6.88

$$\boldsymbol{\gamma = \lim_{n\to\infty}\left(1+\frac{1}{2}+\frac{1}{3}+\cdots+\frac{1}{n}-\log n\right), R_n = 1+\frac{1}{2}+\frac{1}{3}+\cdots+\frac{1}{n}-\log\left(n+\frac{1}{2}\right)}$$

Prove that:

$$\boldsymbol{\frac{1}{23(n+1)^2} < R_n - \gamma < \frac{1}{24n^2}, n \in \mathbb{N}^*}$$

D.W.de Temple

Solution (Omran Kouba)

First, let us define $a_n = \ln\left(n+\frac{1}{2}\right) - \ln\left(n-\frac{1}{2}\right) - \frac{1}{n}$*. Note that*

$$a_n = \int_{-\frac{1}{2}}^{-\frac{1}{2}} \left(\frac{1}{n+t}-\frac{1}{n}\right)dt = -\frac{1}{n}\int_{-\frac{1}{2}}^{-\frac{1}{2}} \frac{t}{t+n}dt = -\frac{1}{n}\left(\int_0^{\frac{1}{2}} \frac{-t}{-t+n}dt + \int_0^{\frac{1}{2}} \frac{t}{t+n}dt\right) =$$

$$= \frac{1}{n}\int_0^{\frac{1}{2}} \frac{2t^2}{n^2-t^2}dt. \text{ So,}$$

$$\frac{1}{n}\int_0^{\frac{1}{2}} \frac{2t^2}{n^2-0}dt < a_n < \frac{1}{n}\int_0^{\frac{1}{2}} \frac{2t^2}{n^2-\frac{1}{4}}dt$$

Equivalently $\frac{1}{12n^3} < a_n < \frac{1}{12n\left(n^2-\frac{1}{4}\right)}$*. Using the trivial inequalities:*

$$\frac{1}{n^2} - \frac{1}{(n+1)^2} < \frac{2}{n^3}, \frac{2}{n\left(n^2-\frac{1}{4}\right)} < \frac{1}{\left(n-\frac{1}{2}\right)^2} - \frac{1}{\left(n+\frac{1}{2}\right)^2}$$

We conclude that $\frac{1}{24}\left(\frac{1}{n^2}-\frac{1}{(n+1)^2}\right)<a_n<\frac{1}{24}\left(\frac{1}{\left(n-\frac{1}{2}\right)^2}-\frac{1}{\left(n+\frac{1}{2}\right)^2}\right)$*. Consequently*

$$\frac{1}{24(n+1)^2}<\sum_{k=n+1}^{\infty}a_k<\frac{1}{24\left(n+\frac{1}{2}\right)^2} \quad (1)$$

Now,

$$\sum_{k=1}^{n}a_k=\ln\left(n+\frac{1}{2}\right)+\ln 2-\sum_{k=1}^{n}\frac{1}{k}$$

So, $\sum_{k=1}^{n}a_k=\ln 2-\gamma$*. Thus,* $\sum_{k=n+1}^{n}a_k\sum_{k=1}^{n}\frac{1}{k}-\ln\left(n+\frac{1}{2}\right)-\gamma$*. Combining this with (1) we get:*

$$\frac{1}{24(n+1)^2}<\sum_{k=1}^{n}\frac{1}{k}-\ln\left(n+\frac{1}{2}\right)-\gamma<\frac{1}{24\left(n+\frac{1}{2}\right)^2} \quad (1)$$

Which is stronger than the proposed inequality.

6.89

$$\boldsymbol{\gamma=\lim_{n\to\infty}\left(1+\frac{1}{2}+\frac{1}{3}+\cdots+\frac{1}{n}-\ln n\right)}$$

Find an increasing order for:

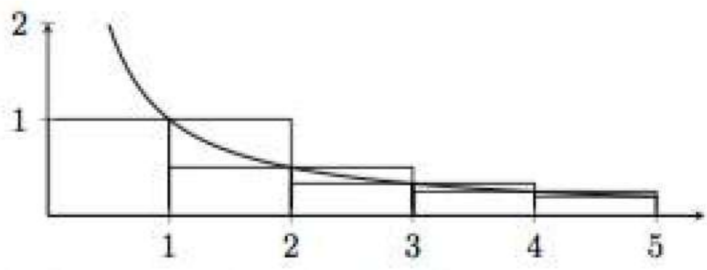
$$\boldsymbol{\Omega_1=\gamma^{\sqrt{\pi e}},\Omega_2=\pi^{\sqrt{e\gamma}},\Omega_3=e^{\sqrt{\gamma\pi}}}$$

Daniel Sitaru

Solution(Emre Tuvay)

From Riemann sum of the area of curve $y=\frac{1}{x}$ *we have the followings for lower bound.*

$$\sum_{k=1}^{n}\frac{1}{k}>\int_{1}^{n+1}\frac{1}{x}dx>\int_{1}^{n}\frac{1}{x}dx=\ln n>0$$



As for upper bound again from Riemann sum keeping $y=\frac{1}{x}$ function's values above the rectangles and adding the area of 1st rectangle we have

$$1+\int_1^n \frac{1}{x}dx > \sum_{k=1}^{n}\frac{1}{k}$$

hence, $0<\gamma<1$.

For convergence, showing the sequence $U_n=\left(1+\frac{1}{2}+\frac{1}{3}+\dots+\frac{1}{n}-\ln n\right)$ *monotonic decreasing should suffice.*

$$U_{n+1}-U_n=\frac{1}{n+1}-\ln(n+1)+\ln n$$

again, by checking the area under $y=\frac{1}{x}$ curve for $x=n$ and $x=n+1$ we see that

$$\int_n^{n+1}\frac{1}{x}dx > \frac{1}{n+1} \Rightarrow \ln(n+1)-\ln n > \frac{1}{n+1}$$

hence,

$U_{n+1}-U_n<0 \Rightarrow U_n=\left(1+\frac{1}{2}+\frac{1}{3}+\dots+\frac{1}{n}-\ln n\right)$ *is monotonic decreasing.*

Therefore,

$\lim_{n\to\infty}\left(1+\frac{1}{2}+\dots+\frac{1}{n}-\ln n\right)$ *converges to γ where $0<\gamma<1$. So,*

$0<\gamma<1<e<\pi$. *Now, for ordering of* $\Omega_1=\gamma^{\sqrt{\pi e}}, \Omega_2=\pi^{\sqrt{e\gamma}}, \Omega_3=e^{\sqrt{\gamma\pi}}$

Considering a generic case, $b^{\sqrt{a}}$ *and* $a^{\sqrt{b}}$ *(where* $a, b \in \Re_{\geq 0}$ *and* $b > a$*) which can be written as* $\left(b^{\frac{1}{\sqrt{b}}}\right)^{\sqrt{a}\sqrt{b}}$ *and* $\left(a^{\frac{1}{\sqrt{a}}}\right)^{\sqrt{b}\sqrt{a}}$ *respectively*

. From checking function, $f(x) = \left(x^{\frac{1}{\sqrt{x}}}\right)$,

$f'(x) = x^{\frac{1}{\sqrt{x}}-1}\left(\frac{1}{\sqrt{x}} - \frac{\ln x}{2\sqrt{x}}\right)$. *Critical point* $f'(x) = \frac{x^{\frac{1}{\sqrt{z}}}}{x^{\frac{3}{2}}}\left(1 - \frac{\ln x}{2}\right) = 0 \Rightarrow x = e^2$.

$f'(x) = \begin{cases} > 0, when\ x < e^2; \\ = 0, when\ x = e^2; \\ < 0, when\ x > e^2; \end{cases}$ *so,* $f(x) = \begin{cases} increasing, when\ x < e^2; \\ maxvalue, when\ x = e^2; \\ decreasing, when\ x > e^2; \end{cases}$ *since,*

$$\gamma < e < \pi < e^2 \Rightarrow f(\gamma) < f(e) < f(\pi) \text{ hence } \Omega_1 < \Omega_3 < \Omega_2$$

6.90 If $a, b, c > 1, n \in \mathbb{N}, n \geq 2$ then:

$$\sum \frac{\sqrt[n]{a^n + 1}}{a^n + 1} + \sum \frac{\sqrt[n]{a^n - 1}}{a^n - 1} > \frac{6}{\sqrt[3]{a^{n-1}b^{n-1}c^{n-1}}}$$

Daniel Sitaru

Solution (Le Van)

With $x > 1$ *and* $n \geq 2$*, building the function:*

$$f(x) = \frac{\sqrt[n]{x+1}}{x+1} - \frac{\sqrt[n]{x}}{x} = (x+1)^{\frac{1}{n}-1} - x^{\frac{1}{n}-1} \Rightarrow$$

$$f'(x) = \left(\frac{1}{n} - 1\right)\left[(x+1)^{\frac{1}{n}-2} - x^{\frac{1}{n}-2}\right] = \left(\frac{1-n}{n}\right)\left[\frac{1}{(x+1)^{\frac{2n-1}{n}}} - \frac{1}{x^{\frac{2n-1}{n}}}\right] > 0$$

Accordingly, $f(x)$ *is a positive function which gives:*

$$f(x) > f(x-1) \Leftrightarrow \frac{\sqrt[n]{x+1}}{x+1} + \frac{\sqrt[n]{x-1}}{x-1} > \frac{2\sqrt[n]{x}}{x} = \frac{2}{\sqrt[n]{x^{n-1}}}$$

Therefore, QED is obtained by AM-GM inequality as:

$$\sum\left(\frac{\sqrt[n]{a^n+1}}{a^n+1} + \frac{\sqrt[n]{a^n-1}}{a^n-1}\right) > \frac{2}{a^{n-1}} + \frac{2}{b^{n-1}} + \frac{2}{c^{n-1}} \geq \frac{6}{\sqrt[3]{a^{n-1}b^{n-1}c^{n-1}}}$$

6.91 If $x, y, z \in \mathbb{R}$ then:

$$\frac{1}{e^{\sin^2 x}}+\frac{1}{e^{\sin^2 y}}+\frac{1}{e^{\sin^2 z}}+\frac{1}{e^{\cos^2 x}}+\frac{1}{e^{\cos^2 y}}+\frac{1}{e^{\cos^2 z}}>3\left(\frac{1}{2}+\frac{\sqrt{e}}{e}\right)$$

Daniel Sitaru

Solution(Marian Ursărescu)

$$\frac{1}{e^{\sin^2 x}}+\frac{1}{e^{\cos^2 x}}\geq 2\sqrt{\frac{1}{e^{\sin^2 x+\cos^2 x}}}=\frac{2}{\sqrt{e}}\Rightarrow \frac{1}{e^{\sin^2 x}}+\frac{1}{e^{\cos^2 x}}\geq \frac{2}{\sqrt{e}}>\frac{1}{2}+\frac{1}{\sqrt{e}}$$

(because $\frac{1}{\sqrt{e}}>\frac{1}{2}\Leftrightarrow$

$$\Leftrightarrow 2>\sqrt{e}\Leftrightarrow 4>e); \left.\begin{array}{l}\frac{1}{e^{\sin^2 y}}+\frac{1}{e^{\cos^2 y}}>\frac{1}{2}+\frac{1}{\sqrt{e}}\\ \frac{1}{e^{\sin^2 z}}+\frac{1}{e^{\cos^2 z}}>\frac{1}{2}+\frac{1}{\sqrt{e}}\end{array}\right\}\Rightarrow \sum\left(\frac{1}{e^{\sin^2 x}}+\frac{1}{e^{\cos^2 x}}\right)>$$

$$3\left(\frac{1}{2}+\frac{1}{\sqrt{e}}\right)$$

6.92 If $x, y, z, t \in \left(0, \frac{\pi}{2}\right)$ then:

$$64\cdot\cos x\cdot\cos z\cdot\sin y\cdot\sin t\cdot\sin(x-y)\cdot\sin(z-t)\leq 1$$

Daniel Sitaru

Solution(Marian Ursărescu)

We must show this:

$$\cos x\cos z\cdot\sin y\cdot\sin t\,(\sin x\cos y-\cos x\sin y)(\sin z\cot t-\cos z\sin t)\leq\frac{1}{64}\quad (1)$$

We show this: $\cos x\sin y\,(\sin x\cos y-\cos x\sin y)\leq\frac{1}{8}$ *(2)*

$$\left.\begin{array}{r}\cos x=a, \sin y=b\ (2)\Leftrightarrow ab\left(\sqrt{(1-a^2)(1-b^2)}-ab\right)\leq\frac{1}{8}\\ But\ \sqrt{(1-a^2)(1-b^2)}\leq\frac{2-a^2-b^2}{2}\end{array}\right\}\Rightarrow$$

$$ab\left(\frac{2-a^2-b^2}{2}-ab\right)\leq\frac{1}{8}\Leftrightarrow ab(2-a^2-b^2-2ab)\leq\frac{1}{4}\Leftrightarrow$$

$$4ab(2-(a+b)^2)\leq 1\quad (3)$$

But $(a+b)^2\geq 4ab\Rightarrow -(a+b)^2\leq -4ab$ *(4)*

From (3)+(4)$\Rightarrow 4ab(2-4ab)\le 1 \Leftrightarrow 8ab-16a^2b^2\le 1 \Leftrightarrow$

$16a^2b^2-8ab+1\ge 0 \Leftrightarrow (4ab-1)^2\ge 0$ *true (equality for* $a=b=\frac{1}{2}$*).*

Similarly: $\cos z\sin t\sin(z-t)\le\frac{1}{8}$ *(5)*

From (2)+(5)$\Rightarrow \cos x\cos z\cdot\sin y\cdot\sin t\cdot\sin(x-y)\sin(z-t)\le 1$*, with*

equality for $x=z=\frac{\pi}{3}$ *and* $y=t=\frac{\pi}{6}$*.*

6.93 Let $n\in\mathbb{N}\wedge n\ge 2$ and $\theta\ge 1$. Prove:

$$\sum_{k=0}^{n}\left(C_n^k\right)^{\theta}>(n+1)\left(\frac{2^n}{n+1}\right)^{\theta}$$

Nguyen Van Nho

Solution (Soumitra Mandal)

We know, $(1+x)^n=1+C_1^nx+C_2^nx^2+\cdots+x^n\Rightarrow 2^n=\sum_{k=0}^{n}C_k^n$

$$\frac{1}{n+1}\sum_{k=1}^{n}(C_k^n)^{\theta}\ge\left(\frac{1}{n+1}\sum_{k=0}^{n}C_k^n\right)^{\theta}=\left(\frac{2^n}{n+1}\right)^{\theta}\Bigg|$$

$$\Rightarrow\sum_{k=0}^{n}(C_k^n)^{\theta}\ge(n+1)\left(\frac{2^n}{n+1}\right)^{\theta}$$

6.94 If $a,b,c,d>0, a+b+c+d=1$ then:

$$\frac{a^2+b+ad}{b+c}+\frac{b^2+c+ba}{c+d}+\frac{c^2+d+cb}{d+a}+\frac{d^2+a+dc}{a+b}\ge 3$$

Marin Chirciu

Solution (Marian Ursărescu)

$$\frac{a^2+b+ad}{b+c}=\frac{a(a+d)+b}{b+c}=\frac{a(1-b-c)+b}{b+c}=\frac{a+b-(b+c)a}{b+c}=$$

$=\frac{a+b}{b+c}-a$ *and similarly* $\Rightarrow$ *we must show:*

$\frac{a+b}{b+c}+\frac{b+c}{c+d}+\frac{c+d}{d+a}+\frac{d+a}{a+b}\ge 4$ *(1)*

$$\frac{a+b}{b+c}+\frac{b+c}{c+d}+\frac{c+d}{d+a}+\frac{d+a}{a+b}=\frac{(a+b)^2}{(a+b)(b+c)}+\frac{(b+c)^2}{(c+d)(b+c)}+\frac{(c+d)^2}{(d+a)(c+d)}+$$

$$+\frac{(d+a)^2}{(a+b)(d+a)}\overset{Bergstrom}{\geq}$$

$$\frac{4(a+b+c+d)^2}{(a+b)(b+c)+(c+d)(b+c)+(d+a)(c+d)+(a+b)(d+a)}=$$

$\frac{4(a+b+c+d)^2}{(a+b+c+d)^2}=4\Rightarrow$ *(1) it is true.*

6.95 If $a,b,c>0$ then:

$$\left(1+\frac{b}{a}+\frac{c}{a}+\frac{a}{a+b+c}\right)\left(1+\frac{c}{b}+\frac{a}{b}+\frac{b}{a+b+c}\right)\left(1+\frac{a}{c}+\frac{b}{c}+\frac{c}{a+b+c}\right)\geq\frac{1000}{27}$$

Daniel Sitaru

Solution (Tran Hong)

$$\left(1+\frac{b}{a}+\frac{c}{a}+\frac{a}{a+b+c}\right)\left(1+\frac{c}{b}+\frac{a}{b}+\frac{b}{a+b+c}\right)\left(1+\frac{a}{c}+\frac{b}{c}+\frac{c}{a+b+c}\right)$$

$$=\frac{[(a+b+c)^2+a^2][(a+b+c)^2][(a+b+c)^2+c^2]}{abc(a+b+c)^3}\overset{(1)}{\geq}\frac{1000}{27}$$

Let $a+b+c=k>0;$

(1) $\Leftrightarrow 27[k^6+(a^2+b^2+c^2)k^4+(a^2b^2+b^2c^2+c^2a^2)k^2+$

$a^2b^2c^2]\geq 1000abck^3$

$$\because a^2+b^2+c^2\geq\frac{(a+b+c)^2}{3}\geq\frac{k^2}{3}$$

$$\because (ab)^2+(bc)^2+(ca)^2\geq abc(a+b+c)=abck$$

We must show that

$27\left(k^6+\frac{k^6}{3}+abck^3+a^2b^2c^2\right)\geq 1000abck^3$ *(2)*

$$\Leftrightarrow 27\left(\frac{4k^6}{3}+abck^3+a^2b^2c^2\right)\geq 1000abck^3$$

$$\Leftrightarrow 36k^6+27a^2b^2c^2\geq 973abck^3\Leftrightarrow 36(k^3-27abc)\left(k^3-\frac{abc}{36}\right)\geq 0$$

It is true because:

$k^3=(a+b+c)^3\geq 27abc>\frac{abc}{36}\Rightarrow$ *(2) true* $\Rightarrow$ *(1) true.*

6.96 If $x, y, z \geq 0$ then:

$$\pi^{x^2(x^2+1)} + \pi^{y^2(y^2+1)} + \pi^{z^2(z^2+1)} \geq \pi^{xy(xy+1)} + \pi^{yz(yz+1)} + \pi^{zx(zx+1)}$$

Daniel Sitaru

Solution (Klevis Liperi)

$$\pi^{x^2(x^2+1)} + \pi^{y^2(y^2+1)} \overset{AM-GM}{\geq} 2\pi^{\frac{x^2(x^2+1)+y^2(y^2+1)}{2}} \overset{AM-GM}{\geq}$$

$$\geq 2\pi^{\sqrt{x^2y^2(x^2+1)(y^2+1)}} = 2\pi^{xy\sqrt{(x^2+1)(y^2+1)}} \overset{CBS}{\geq} 2\pi^{xy(xy+1)} \quad (1)$$

In the same way, we can prove:

$$\pi^{y^2(y^2+1)} + \pi^{z^2(z^2+1)} \geq 2\pi^{yz(yz+1)} \quad (2)$$

And $\pi^{x^2(x^2+1)} + \pi^{z^2(z^2+1)} \geq 2\pi^{zx(zx+1)}$ *(3)*

(1)+(2)+(3) gives the desired inequality. Equality holds if $x = y = z$.

6.97 If $a, b \geq 0$ then:

$$(a^2 + b^2)\left(a + b + \sqrt{a^2 + b^2}\right)^2 \geq 18a^2b^2$$

Daniel Sitaru

Solution(Elvin Samedov)

I will prove a stronger inequality than Dan Sitaru's inequality:

If $\{a, b\} \geq 0 \quad (a^2 + b^2)\left(a + b + \sqrt{a^2 + b^2}\right)^2 \geq 18a^2b^2$ *(1)*

Proposed by Dan Sitaru

If $\{a, b\} \geq 0 \quad (a^2 + b^2)\left(a + b + \sqrt{a^2 + b^2}\right)^2 \geq 4\left(\sqrt{2} + 1\right)^2 a^2b^2$ *(2)*

Proposed by E. Samedov

It is enough to prove inequality (2)

Let $\sqrt{a^2 + b^2} = m, ab = n$. *We must prove*

(2) $\Rightarrow \sqrt{a^2 + b^2}\left(a + b + \sqrt{a^2 + b^2}\right) \geq 2\left(\sqrt{2} + 1\right)ab \Rightarrow$

$$m\left(\sqrt{m^2 + 2n} + m\right) \geq 2\left(\sqrt{2} + 1\right)n$$

$$\Rightarrow \sqrt{m^4+2m^2n}+m^2-2(\sqrt{2}+1)n\geq 0$$

$$\begin{cases} m^2\geq 2n \\ m^4\geq 4n^2 \\ 2m^2n\geq 4n^2 \end{cases}\Rightarrow\begin{cases}\sqrt{m^4+2m^2n}\geq 2\sqrt{2}n \\ m^2\geq 2n\end{cases}\Rightarrow$$

$$\sqrt{m^4+2m^2n}+m^2\geq 2n+2\sqrt{2}n$$

$$\Rightarrow \sqrt{m^4+2m^2n}+m^2-2(1+\sqrt{2})n\geq 0\Leftrightarrow$$

$$\Leftrightarrow (a^2+b^2)\left(a+b+\sqrt{a^2+b^2}\right)^2\geq 4(\sqrt{2}+1)^2a^2b^2$$

6.98 If $a,b,c\in\mathbb{R}$, $a^2+b^2+c^2\leq 15$ then:

$$\frac{1}{\sqrt{4+a^2}}+\frac{1}{\sqrt{4+b^2}}+\frac{1}{\sqrt{4+c^2}}\geq 1$$

Daniel Sitaru

Solution(Abdallah Al Farissi)

$f(x)=\frac{1}{\sqrt{4+x}}$ *is convexe function then:*

$$f(a^2)+f(b^2)+f(c^2)\geq 3f\left(\frac{a^2+b^2+c^2}{3}\right)=3f(5)=1 \text{ then:}$$

$$\frac{1}{\sqrt{4+a^2}}+\frac{1}{\sqrt{4+b^2}}+\frac{1}{\sqrt{4+c^2}}\geq 1$$

6.99 If $x,y>0$ then:

$$4\left(x+\frac{x+1}{y}\right)\left(y+\frac{y+1}{x}\right)\leq\left(2+x+y+\frac{1}{x}+\frac{1}{y}\right)^2$$

Andrei Ștefan Mihalcea

Solution (Ravi Prakash)

Consider

$$\left(2+x+y+\frac{1}{x}+\frac{1}{y}\right)^2-4\left(x+\frac{x+1}{y}\right)\left(y+\frac{y+1}{x}\right)=$$

$$=4+(x+y)^2+\left(\frac{x+y}{xy}\right)^2+4(x+y)+4\left(\frac{x+y}{xy}\right)+\frac{2(x+y)^2}{xy}-$$

$$-4\left[xy+x+1+y+1+\frac{(x+1)(y+1)}{xy}\right]$$

$$=(x+y)^2-4xy+\left(\frac{x+y}{xy}\right)^2-4\left(\frac{1}{x}+\frac{1}{y}\right)+4\left(\frac{1}{x}+\frac{1}{y}\right)-\frac{4}{xy}+$$

$$+4(x+y)-4(x+y)+4-12+\frac{2(x+y)^2}{xy}$$

$$=(x-y)^2+\frac{(x-y)^2}{x^2y^2}+\frac{2(x-y)^2}{xy}\geq 0$$

6.100 If $x,y,z>0$ then:

$$\frac{e^{x^3+y^3}}{e^{2(x+y)}}+\frac{e^{y^3+z^3}}{e^{2(y+z)}}+\frac{e^{z^3+x^3}}{e^{2(z+x)}}\geq\frac{1}{e^2}(x^xy^y+y^yz^z+z^zx^x)$$

Daniel Sitaru

Solution(Soumitra Mandal)

We know, $\frac{x}{1+x}\leq\ln(1+x)\leq x$, *replacing* x *by* $x-1$

$$\frac{x-1}{x}\leq\ln x\leq x-1.$$

Let $f(x)=x^3-2x-x\ln x+1\geq x^3-2x-x(x-1)+1$

$=x^3-x^2-x+1=(x+1)(x-1)^2\geq 0$. *Hence* $f(x)\geq 0$ *for all* $x\geq 1$

Hence $x^3-2x\geq x\ln x-1$

$$\therefore\sum_{cyc}e^{x^3+y^3-2(x+y)}\geq\sum_{cyc}e^{x\ln x-1+y\ln y-1}$$

$$\therefore\sum_{cyc}\frac{e^{x^3+y^3}}{e^{2(x+y)}}\geq\frac{1}{e^2}\sum_{cyc}x^xy^y$$

6.101 Let $a,b,c\in(0;+\infty)$. Prove:

$$\left(\frac{a^2-ab+b^2}{b^2+bc+c^2}\right)^3+\left(\frac{b^2-bc+c^2}{c^2+ca+a^2}\right)^3+\left(\frac{c^2-ca+a^2}{a^2+ab+b^2}\right)^3\geq\frac{1}{9}$$

Nguyen Van Nho

Solution(Do Duc Huy)

$$\text{Let } a,b,c>0$$

$$\text{We have: } a^2-ab+b^2\geq\frac{1}{3}(a^2+ab+b^2)$$

$$\Leftrightarrow 2(a-b)^2\geq 0\ \ (\text{true})$$

$$\text{So, }\sum\left(\frac{a^2-ab+b^2}{b^2+bc+c^2}\right)^3\geq 3\cdot\frac{a^2-ab+b^2}{b^2+bc+c^2}\cdot\frac{b^2-bc+c^2}{c^3+ca+a^2}\cdot\frac{c^2-ca+a^2}{a^2+ab+b^2}$$

$$\geq 3\cdot\frac{1}{3}\cdot\frac{a^2+ab+b^2}{b^2+bc+c^2}\cdot\frac{1}{3}\cdot\frac{b^2+bc+c^2}{c^2+ca+a^2}\cdot\frac{c^2+ca+a^2}{a^2+ab+b^2}\cdot\frac{1}{3}=\frac{1}{9}\Rightarrow Q.E.D.$$

$$\text{“}=\text{”}\Leftrightarrow a=b=c$$

6.102 If $a,b,c>0$ then:

$$\sum_{cyc}\left(\left(\frac{2\sqrt{ab}}{a+b}\right)^{\frac{a+b}{2\sqrt{ab}}}+\frac{a+b}{2\sqrt{ab}}\right)\geq 6$$

Daniel Sitaru

Solution (Soumava Chakraborty)

$$\left(\frac{2\sqrt{ab}}{a+b}\right)^{\frac{a+b}{2\sqrt{ab}}}=\left(1+\left(\frac{2\sqrt{ab}}{a+b}-1\right)\right)^{\frac{a+b}{2\sqrt{ab}}}$$

$$\overset{Bernoulli}{\geq}1+\left(\frac{2\sqrt{ab}}{a+b}-1\right)\left(\frac{a+b}{2\sqrt{ab}}\right)$$

$$\left[\because\frac{2\sqrt{ab}}{a+b}-1>-1\ \&\ \frac{a+b}{2\sqrt{ab}}\overset{A-G}{\geq}1\right]$$

$$=2-\frac{a+b}{2\sqrt{ab}}\Rightarrow\left(\frac{2\sqrt{ab}}{a+b}\right)^{\frac{a+b}{2\sqrt{ab}}}\overset{(1)}{\geq}2-\frac{a+b}{2\sqrt{ab}}$$

$$\text{Similarly, }\left(\frac{2\sqrt{bc}}{b+c}\right)^{\frac{b+c}{2\sqrt{bc}}}\overset{(2)}{\geq}2-\frac{b+c}{2\sqrt{bc}}\ \&\ \left(\frac{2\sqrt{ca}}{c+a}\right)^{\frac{c+a}{2\sqrt{ca}}}\overset{(3)}{\geq}2-\frac{c+a}{2\sqrt{ca}}$$

$$(1)+(2)+(3)\Rightarrow LHS\geq 6-\sum\frac{a+b}{2\sqrt{ab}}+\sum\frac{a+b}{2\sqrt{ab}}=6$$

6.103 If $a,b,c>0, a+b+c=6$ then:

$$\frac{31-6b-6c}{a^2+b+c-5}+\frac{31-6c-6a}{b^2+c+a-5}+\frac{31-6a-6b}{c^2+a+b-5}\le 7$$

Iuliana Trașcă

Solution (Șerban George Florin)

$$\sum_{a,b,c}\frac{31-6b-6c}{a^2+b+c-5}=\sum_{a,b,c}\frac{31-6(b+c)}{a^2+b+c-5}=\sum_{a,b,c}\frac{31-6(6-a)}{a^2+6-a-5}=$$

$$\sum\frac{6a-5}{a^2-a+1}, \begin{matrix}a+b+c=6\\ a,b,c>0\end{matrix}, \begin{matrix}a+b=6-c>0\\ \Rightarrow c<6\end{matrix}$$

$$b+c=6-a>0\Rightarrow a<6, a+c=6-b>0\Rightarrow b<6\Rightarrow a,b,c<0$$

$$\Rightarrow a,b,c\in(0,6)$$

$$\frac{6a-5}{a^2-a+1}\le-\frac{a+9}{3}, (\forall)a\in(0,6)$$

$$a^2-a+1>0, \Delta=-3<0, (\forall)a\in\mathbb{R}\Rightarrow 18a-15$$

$$\le(-a+9)(a^2-a+1)$$

$$18a-15\le-a^3+a^2-a+9a^2-9a+9$$

$$-a^3+10a^2-28a+24\ge0$$

$$-a^3+2a^2+8a^2-16a-12a+24\ge0$$

$$-a^2(a-2)+8a(a-2)-12(a-2)\ge0$$

$$(a-2)(-a^2+8a-12)\ge0$$

$$(a-2)(-a^2+2a+6a-12)\ge0$$

$$(a-2)[-a(a-2)+6(a-2)]\ge0$$

$$(a-2)(a-2)(6-a)\ge0, (a-2)^2(6-a)\ge0$$

$$(a-2)^2\ge0, 6-a>0\Rightarrow a<6, (\forall)a\in(0,6)\ \textit{true}$$

$$\Rightarrow\sum\frac{31-6b-6c}{a^2+b+c-5}\le\sum\frac{-a+9}{3}=-\frac{1}{3}\sum a+\sum 3=$$

$$=-\frac{1}{3}\cdot6+9=-2+9=7\ \textit{true.}$$

6.104 If $a,b,c>1, m,n>0$ then:

$$\frac{\log_a^2 b}{m\log_b c+n\log_c a}+\frac{\log_b^2 c}{m\log_c a+n\log_a b}+\frac{\log_c^2 a}{m\log_a b+n\log_b c}\geq\frac{3}{m+n}$$

D.M.Batinetu-Giurgiu, Neculai Stanciu

Solution(Lazaros Zachariadis)

$$LHS=\frac{\log_a^2 b}{m\cdot\log_b c+n\cdot\log_c a}+\frac{\log_b^2 c}{m\cdot\log_c a+n\cdot\log_a b}$$
$$+\frac{\log_c^2 a}{m\cdot\log_a b+n\cdot\log_b c}\overset{Bergstrom}{\geq}$$
$$\geq\frac{(\log_a b+\log_b c+\log_c a)^2}{m(\log_b c+\log_c a+\log_a b)+n(\log_c a+\log_a b+\log_b c)}$$
$$=\frac{\frac{\ln b}{\ln a}+\frac{\ln c}{\ln b}+\frac{\ln a}{\ln c}}{m+n}\overset{AM-GM}{\geq}\frac{3\sqrt[3]{\frac{\ln b}{\ln a}\cdot\frac{\ln c}{\ln b}\cdot\frac{\ln a}{\ln c}}}{m+n}=\frac{3}{m+n}=RHS$$

6.105 Let $a,b,c\in(0;+\infty)\wedge a^3+b^3+c^3=3$. Prove:

$$\left(a+2\sqrt{b}+\sqrt[3]{c}\right)\left(b+2\sqrt{c}+\sqrt[3]{a}\right)\left(c+2\sqrt{a}+\sqrt[3]{b}\right)\leq 64$$

Nguyen Van Nho

Solution (Marian Ursărescu)

We must show: $\sqrt[3]{\left(a+2\sqrt{b}+\sqrt[3]{c}\right)\left(b+2\sqrt{c}+\sqrt[3]{a}\right)\left(c+2\sqrt{a}+\sqrt[3]{b}\right)}\leq 4$ *(1)*

But

$$\sqrt[3]{\left(a+2\sqrt{b}+\sqrt[3]{c}\right)\left(b+2\sqrt{c}+\sqrt[3]{a}\right)\left(c+2\sqrt{a}+\sqrt[3]{b}\right)}\leq$$
$$\frac{a+b+c+2\left(\sqrt{a}+\sqrt{b}+\sqrt{c}\right)+\sqrt[3]{a}+\sqrt[3]{b}+\sqrt[3]{c}}{3}\quad (2)$$

Form (1)+(2) we must show:

$$a+b+c+2\left(\sqrt{a}+\sqrt{b}+\sqrt{c}\right)+\sqrt[3]{a}+\sqrt[3]{b}+\sqrt[3]{c}\leq 12\quad (3)$$

From Hölder's inequality we have: $a^3+b^3+c^3\geq\frac{(a+b+c)^3}{9}\Leftrightarrow$

$$(a+b+c)^3 \le 27 \Leftrightarrow a+b+c \le 3 \ \ (4)$$

From Cauchy's inequality: $\left(\sqrt{a}+\sqrt{b}+\sqrt{c}\right)^2 \le 3(a+b+c) \le 9 \Rightarrow$

$$\sqrt{a}+\sqrt{b}+\sqrt{c} \le 3 \ \ (5)$$

Again, from Hölder's inequality $\Rightarrow$

$$\frac{\left(\sqrt[3]{a}+\sqrt[3]{b}+\sqrt[3]{c}\right)^3}{9} \le a+b+c \le 3 \Rightarrow \sqrt[3]{a}+\sqrt[3]{b}+\sqrt[3]{c} \le 3 \ \ (6)$$

From (4)+(5)+(6)$\Rightarrow$

$$a+b+c+2\left(\sqrt{a}+\sqrt{b}+\sqrt{c}\right)+\sqrt[3]{a}+\sqrt[3]{b}+\sqrt[3]{c} \le 12 \Rightarrow \text{(3) is true.}$$

6.106 Let $a,b,c,d \in (0;+\infty) \wedge abcd = 1$. Prove:

$$\frac{d}{a^5+b^5+c^5+d}+\frac{c}{a^5+b^5+d^5+c}+\frac{b}{a^5+d^5+c^5+b}+\frac{a}{d^5+b^5+c^5+a} \le 1$$

Nguyen Van Nho

Solution (Soumava Chakraborty)

$$a^5+b^5+c^5+d \overset{Chebyshev}{\ge} \frac{1}{3}(a^2+b^2+c^2)(a^3+b^3+c^3)+d$$

$$\overset{A-G}{\ge} abc(a^2+b^2+c^2)+d \overset{abcd=1}{=} abc(a^2+b^2+c^2)+d(abcd)$$

$$= abc(a^2+b^2+c^2+d^2) = \frac{a^2+b^2+c^2+d^2}{d} \ \left(\because abc = \frac{1}{d}\right)$$

$$\Rightarrow \frac{d}{a^5+b^5+c^5+d} \overset{(1)}{\le} \frac{d^2}{a^2+b^2+c^2+d^2}$$

Similarly, $\frac{a}{b^5+c^5+d^5+a} \overset{(2)}{\le} \frac{a^2}{a^2+b^2+c^2+d^2}$

$$\frac{b}{c^5+d^5+a^5+b} \overset{(3)}{\le} \frac{b^2}{a^2+b^2+c^2+d^2} \ \ \& \ \frac{c}{d^5+a^5+b^5+c} \overset{(4)}{\le} \frac{c^2}{a^2+b^2+c^2+d^2}$$

(1)+(2) +(3) +(4)$\Rightarrow LHS \le \frac{\sum a^2}{\sum a^2} = 1$ *(proved)*

6.107 Let $a, b, c > 0$ and $\sum ab = 1$. Show that:

$$4 + 3\sum a^3b \le \sum a^2 + 4\left(\sum a\right)\left(\sum a^2b\right)$$

Andrei Ștefan Mihalcea

Solution (Șerban George Florin)

$$\sum a \cdot \sum a^2b = (a+b+c)(a^2b+b^2c+c^2a) = a^3b +$$

$$+ab^2c + a^2c^2 + a^2b^2 + b^3c + abc^2 + a^2bc + b^2c^2 + a^2c^2$$

$$= \sum a^3b + \sum a^2b^2 + abc(a+b+c)$$

$$\sum ab = 1 \Rightarrow \left(\sum ab\right)^2 = 1^2, \sum a^2b^2 + 2abc(a+b+c) = 1$$

$$\Rightarrow \sum a^2b^2 = 1 - 2abc(a+b+c)$$

$$\sum a \cdot \sum a^2b = \sum a^3b + 1 - 2abc(a+b+c) + abc(a+b+c) =$$

$$= \sum a^3b + 1 - abc(a+b+c)$$

$$4 + 3\sum a^3b \le \sum a^2 + 4\left(\sum a\right)\cdot\left(\sum a^2b\right)$$

$$\Rightarrow 4 + 3\sum a^3b \le \sum a^2 + 4\sum a^3b + 4 - 4abc(a+b+c)$$

$$\Rightarrow \sum a^2 + \sum a^3b \ge 4abc(a+b+c)$$

$$\sum a^2(1+ab) \ge 4abc(a+b+c)$$

$$ab + bc + ac = 1 \overset{Ma\ge Mg}{\ge} 3\sqrt[3]{a^2b^2c^2} \Rightarrow 1 \ge 27a^2b^2c^2 \Rightarrow abc \le \frac{1}{3\sqrt{3}}$$

$$\sum a^2(1+ab) = \sum a^2 + \sum a^3b \ge 4abc(a+b+c)$$

$$\sum a^3b = \sum \frac{a^2}{\frac{1}{ab}} \overset{Bergstrom}{\ge} \frac{(a+b+c)^2}{\frac{1}{ab}+\frac{1}{ac}+\frac{1}{bc}} = \frac{abc(a+b+c)^2}{a+b+c}$$

$$= abc(a+b+c)$$

$$\sum a^2 \geq 3abc(a+b+c), (a+b+c)^2 \leq 3(a^2+b^2+c^2)$$

$$\Rightarrow \sum a^2 \geq \frac{(a+b+c)^2}{3} \geq 3abc(a+b+c) \Rightarrow (a+b+c)^2 \geq 9abc(a+b+c)$$

$$\Rightarrow a+b+c \geq 9abc, a+b+c \overset{Ma\geq Mg}{\geq} 3\sqrt[3]{abc} \geq 9abc$$

$$3\sqrt[3]{abc} \geq 9abc, \sqrt[3]{abc} \geq 3abc, abc \geq 27a^3b^3c^3$$

$$\Rightarrow a^2b^2c^2 \leq \frac{1}{27}, abc \leq \frac{1}{3\sqrt{3}}, \textit{true}$$

$$\Rightarrow \sum a^2(1+ab) = \sum a^2 + \sum a^3b \geq 3abc(a+b+c) + abc(a+b+c) = 4abc(a+b+c)$$

6.108 If $a, b, c > 0, a+b+c=1$ then:

$$8\prod(1+ab-c) \leq \prod(1+2a-a^2)$$

Andrei Ștefan Mihalcea

Solution (Sanong Huayrerai)

Because $a, b, c > 0$ and $a+b+c=1$, we have:

$$8(1+ab-c)(1+bc-a)(1+ca-b) \leq (1+2a+a^2)(1+2b-b^2)(1+2c-c^2)$$

$$\textit{If } 8(a+b+ab)(b+c+bc)(c+a+ca) \leq$$

$$\leq \big(1+a+a(1-a)\big)\big(1+b+b(1-b)\big)\big(1+c+c(1-c)\big)$$

$$\textit{If } \big(2(a+b+ab)\big)\big(2(b+c+bc)\big)\big(2(c+a+ca)\big) \leq$$

$$\leq (1+a+ab+ac)(1+b+ab+bc)(1+c+bc+ac)$$

$$= (a+b+ab+a+c+ac)(a+b+ab+b+c+bc)(b+c+bc+c+a+ca)$$

If $(2x)(2y)(2z) \leq (x+y)(y+z)(z+x)$, where

$a+b+ab: x, b+c+bc: y$ and $c+a+ca: z$

and it is true.

$$Because\ (x+y)(y+z)(z+x)=$$

$$x^2y+x^2z+y^2x+y^2z+z^2x+z^2y+2xyz\geq 8xyz$$

6.109 If $x,y,z>0$ then:

$$\sum_{cyc}\frac{x}{y}+4\sum_{cyc}\frac{y}{x+y}+4\sum_{cyc}\frac{xy}{(x+y)^2}\geq 12$$

Daniel Sitaru

Solution (Tran Hong)

$$Must\ show\ that: \frac{x}{y}+4\cdot\frac{y}{x+y}+4\cdot\frac{xy}{(x+y)^2}\geq 4\ \ (etc)$$

$$\Leftrightarrow x(x+y)^2+4y^2(x+y)+4xy^2\geq 4y(x+y)^2$$

$$\Leftrightarrow x^3-2x^2y+xy^2\geq 0$$

$$\Leftrightarrow x(x^2-2xy+y^2)\geq 0\Leftrightarrow x(x-y)^2\geq 0\ \ (true)$$

$$\Rightarrow\sum\frac{x}{y}+4\sum\frac{y}{x+y}+4\sum\frac{xy}{(x+y)^2}\geq 4+4+4=12$$

6.110 If $a,b,c>0$ then:

$$\frac{e^{3a}+e^{3b}+e^{3c}+3e^{a+b+c}}{e^{3\sqrt{ab}}+e^{3\sqrt{bc}}+e^{3\sqrt{ca}}}\geq 2$$

Daniel Sitaru

Solution(Tran Hong)

$$Let\ f(x)=e^{3x}(x>0)\Rightarrow f'(x)=3e^{3x}\Rightarrow f''(x)=9e^{3x}>0\ (\forall x>0)$$

Using T. Popoviciu's inequality we have:

$$f(a)+f(b)+f(c)+3f\left(\frac{a+b+c}{3}\right)\geq$$

$$2\left[f\left(\frac{a+b}{2}\right)+f\left(\frac{b+c}{2}\right)+f\left(\frac{c+a}{2}\right)\right]$$

$$\Leftrightarrow e^{3a}+e^{3b}+e^{3c}+3e^{3\cdot\frac{a+b+c}{3}}\geq 2\left[e^{3\cdot\frac{a+b}{2}}+e^{3\cdot\frac{b+c}{2}}+e^{3\cdot\frac{c+a}{2}}\right]=2\cdot\sum e^{3\cdot\frac{a+b}{2}}$$

But:

$$\frac{a+b}{2}\overset{AM-GM}{\geq}\sqrt{ab}\ (etc)\Rightarrow 2\cdot\sum e^{3\cdot\frac{a+b}{2}}\geq 2\cdot\sum e^{3\sqrt{ab}}$$

$$\Rightarrow e^{3a}+e^{3b}+e^{3c}+3e^{a+b+c}\geq 2\sum e^{3\sqrt{ab}}$$

6.111 If $x,y,z>0, xyz=1$ then:

$$\sum_{cyc}\left(\left(\frac{x}{y}+\frac{y}{x}\right)z+\frac{16}{(x+y)^2}-6x\right)\geq 0$$

Daniel Sitaru

Solution(Tran Hong)

$$\sum\left[\left(\frac{x}{y}+\frac{y}{x}\right)z+\frac{16}{(x+y)^2}-6x\right]\geq 0$$

$$\Leftrightarrow\left[\left(\frac{x}{y}+\frac{y}{x}\right)z+\frac{16}{(x+y)^2}-6x\right]+\left[\left(\frac{y}{z}+\frac{z}{y}\right)+\frac{16}{(y+z)^2}-6y\right]+$$

$$+\left[\left(\frac{x}{z}+\frac{z}{x}\right)y+\frac{16}{(z+x)^2}-6z\right]\geq 0$$

$$\Leftrightarrow\left(\frac{x}{y}+\frac{y}{x}\right)z+\frac{16}{(x+y)^2}-6z+\left(\frac{y}{z}+\frac{z}{y}\right)x+\frac{16}{(y+z)^2}-6x+\left(\frac{x}{z}+\frac{z}{x}\right)y+$$

$$+\frac{16}{(z+x)^2}-6y\geq 0$$

$$\Leftrightarrow\sum\left[\left(\frac{x}{y}+\frac{y}{x}\right)z+\frac{16}{(x+y)^2}-6z\right]\geq 0$$

We must show that:

$$\left(\frac{x}{y}+\frac{y}{x}\right)z+\frac{16}{(x+y)^2}-6z\geq 0\ \ (etc)$$

$$\Leftrightarrow\frac{x^2+y^2}{(xy)^2}+\frac{16}{(x+y)^2}-\frac{6}{xy}\geq 0\ \ (\because xyz=1)$$

$$\Leftrightarrow(x^2+y^2)(x+y)^2+16(xy)^2-6xy(x+y)^2\geq 0$$

$$\Leftrightarrow x^4-4x^3y+6x^2y^2-4xy^3+y^4\geq 0\Leftrightarrow(x-y)^4\geq 0\ \ (true)$$

Equality $\Leftrightarrow x=y=z=1.$

6.112 If $x, y, z > 0, \sqrt{xy}+\sqrt{yz}+\sqrt{zx}=3$ then:

$$\left(x-\sqrt{x}\right)+\left(y-\sqrt{y}\right)+\left(z-\sqrt{z}\right)+2\left(\sqrt[4]{xyz}-1\right)\geq 0$$

Daniel Sitaru

Solution (Tran Hong)

Let $u=\sqrt{x}; v=\sqrt{y}; w=\sqrt{z}\ (u,v,w>0)$

$$\Rightarrow uv+vw+wu=3; x=u^2; y=v^2; z=w^2$$

We must show that: $u^2+v^2+w^2-(u+v+w)+2\left(\sqrt{uvw}-1\right)\geq 0$

$$\Leftrightarrow [(u+v+w)^2-6]-(u+v+w)+2\sqrt{uvw}-2\geq 0$$

$$\Leftrightarrow (u+v+w)^2-(u+v+w)+2\sqrt{uvw}-8\geq 0$$

$$\Leftrightarrow 2\sqrt{uvw}\geq 8+(u+v+w)-(u+v+w)^2$$

(Let $p=u+v+w; q=uv+vw+wu=3; r=uvw$*)*

$$\Leftrightarrow 2\sqrt{r}\geq 8+p-p^2 \quad (*)$$

$\because$ *We have:* $p=u+v+w\geq\sqrt{3(uv+vw+wu)}=\sqrt{3\cdot 3}=3$

If $p\geq\frac{1}{2}\left(1+\sqrt{33}\right)>3\Rightarrow 8+p-p^2\leq 0<2\sqrt{r}\Rightarrow$ *(*) true*

If $3\leq p<\frac{1}{2}\left(1+\sqrt{33}\right)$ *then:* $r\geq\frac{\left(8+p-p^2\right)^2}{4}$*. But by Schur's we have:*

$r\geq\frac{p\left(4q-p^2\right)}{9}=\frac{p\left(12-p^2\right)}{9}$*. We need to prove:* $\frac{p\left(12-p^2\right)}{9}\geq\frac{\left(8+p-p^2\right)^2}{4}$

$$\Leftrightarrow (p-3)\left(p^3+\frac{13}{9}p^2-\frac{32}{3}p-\frac{64}{3}\right)\leq 0.$$

It is true because:

$$3\leq p<\frac{1}{2}\left(1+\sqrt{33}\right)$$

$$\Rightarrow\begin{cases} p-3\geq 0 \\ p^3+\frac{13}{9}p^2-\frac{32}{3}p-\frac{64}{3}<\left[\frac{1}{2}\left(1+\sqrt{33}\right)\right]^3+\frac{13}{9}\left[\frac{1}{2}\left(1+\sqrt{33}\right)\right]^2-96\approx -52<0\end{cases}$$

$\Rightarrow$ *(*) true* $\Rightarrow$ *proved.*

6.113 If $x, y, z > 0$ then:

$$x^x + y^y + z^z \geq \sqrt{\left(\frac{x+y}{2}\right)^{x+y}} + \sqrt{\left(\frac{y+z}{2}\right)^{y+z}} + \sqrt{\left(\frac{z+x}{2}\right)^{z+x}}$$

Daniel Sitaru

Solution(Dinh Thuan Tien)

$$x^x + y^y + z^z \geq \sqrt{\left(\frac{x+y}{2}\right)^{x+y}} + \sqrt{\left(\frac{y+z}{2}\right)^{y+z}} + \sqrt{\left(\frac{x+z}{2}\right)^{x+z}}, (x, y, z > 0)$$

$$f(x) = x^x \Rightarrow f''(x) = x^x(logx + 1)^2 + \frac{1}{x}x^x > 0$$

$$x^x + y^y \geq 2\left(\frac{x+y}{2}\right)^{\frac{x+y}{2}} = 2\sqrt{\left(\frac{x+y}{2}\right)^{x+y}} \quad (1)$$

$$y^y + z^z \geq 2\sqrt{\left(\frac{y+z}{2}\right)^{y+z}} \ (2)$$

$$x^x + z^z \geq 2\sqrt{\left(\frac{x+z}{2}\right)^{x+z}} \ (3)$$

$$x^x + y^y + z^z \geq \sqrt{\left(\frac{x+y}{2}\right)^{x+y}} + \sqrt{\left(\frac{y+z}{2}\right)^{y+z}} + \sqrt{\left(\frac{x+z}{2}\right)^{x+z}}$$

6.114 If $x, y, z > 0$ then:

$$\frac{2xy}{(x+y)^2} + \frac{2xz}{(x+z)^2} + \frac{2yz}{(y+z)^2} \leq \frac{1}{2} + \frac{8xyz}{(x+y)(y+z)(z+x)}$$

Rahim Shahbazov

Solution(Tran Hong)

$$\frac{2xy}{(x+y)^2} + \frac{2xz}{(x+z)^2} + \frac{2yz}{(y+z)^2} \leq \frac{1}{2} + \frac{8xyz}{(x+y)(y+z)(z+x)}$$

$$\Leftrightarrow \frac{-4xy}{(x+y)^2} + \frac{-4xz}{(x+z)^2} + \frac{-4yz}{(y+z)^2} \geq \frac{-16xyz}{(x+y)(y+z)(z+x)} - 1$$

$$\Leftrightarrow 1-\frac{4xy}{(x+y)^2}+1-\frac{4xz}{(x+z)^2}+1-\frac{4yz}{(y+z)^2}+\frac{16xyz-2(x+y)(y+z)(z+x)}{(x+y)(y+z)(z+x)}\ge 0$$

$$\Leftrightarrow \frac{(x-y)^2}{(x+y)^2}+\frac{(x-z)^2}{(x+z)^2}+\frac{(y-z)^2}{(y+z)^2}-\frac{2x(y-z)^2+2y(x-z)^2+2z(x-y)^2}{(x+y)(y+z)(z+x)}\ge 0$$

$$\Leftrightarrow \sum_{cyc}\frac{(y-z)^2}{(y+z)^2}\left(\frac{1}{y+z}-\frac{2x}{(x+y)(x+z)}\right)\ge 0$$

$$\Leftrightarrow \sum_{cyc}\left(\frac{(y-z)^2}{(y+z)^2}\cdot\frac{(x-y)(x-z)}{(x+y)(y+z)(z+x)}\right)\ge 0$$

$$\Leftrightarrow -\frac{(x-y)(y-z)(z-x)}{(x+y)(y+z)(z+x)}\cdot\frac{(x-y)(y-z)(x-z)}{(x+y)(y+z)(z+x)}\ge 0$$

$$\Leftrightarrow \left(\frac{(x-y)(y-z)(z-x)}{(x+y)(y+z)(z+x)}\right)^2\ge 0$$

Which is true.Equality$\Leftrightarrow x=y=z$

Let: $x=a;y=b;z=c\Rightarrow(*)$ *is true.Proved.*

6.115 If $a,b,c>0,abc=1$ then:

$$\sqrt[4]{\frac{b^2+c^2}{2a}}+\sqrt[4]{\frac{c^2+a^2}{2b}}+\sqrt[4]{\frac{a^2+b^2}{2c}}\le a+b+c$$

George Apostolopoulos

Solution(Rahim Shahbazov)

$$\sqrt[4]{\frac{b^2+c^2}{2a}}+\sqrt[4]{\frac{c^2+a^2}{2b}}+\sqrt[4]{\frac{a^2+b^2}{2c}}\le a+b+c \quad (1)$$

$$\frac{b^2+c^2}{2a}=\frac{bc(b^2+c^2)}{2}=\frac{8bc(b^2+c^2)}{16}=\frac{4(2bc)(b^2+c^2)}{16}\le\frac{(b+c)^4}{16}\overset{(1)}{\Rightarrow}$$

$$LHS\le\frac{b+c}{2}+\frac{c+a}{2}+\frac{a+b}{2}=a+b+c$$

6.116 If $a, b, c > 0$ then:

$$\frac{b+c}{a}+\frac{c+a}{b}+\frac{a+b}{c}\geq 2\cdot\sqrt{(ab+bc+ca)\left(\frac{1}{a^2}+\frac{1}{b^2}+\frac{1}{c^2}\right)}$$

When does equality holds?

Nguyen Van Canh

Solution (Tran Hong)

$$\frac{b+c}{a}+\frac{c+a}{b}+\frac{a+b}{c}\geq 2\cdot\sqrt{(ab+bc+ca)\left(\frac{1}{a^2}+\frac{1}{b^2}+\frac{1}{c^2}\right)}\Leftrightarrow$$

$$\left(\frac{b+c}{a}+\frac{c+a}{b}+\frac{a+b}{c}\right)^2\geq 4\left(\sum_{cyc}ab\cdot\sum_{cyc}\frac{1}{a^2}\right)\Leftrightarrow$$

$$\big(ab(a+b)+bc(b+c)+ca(c+a)\big)\geq 4\sum_{cyc}a\cdot\sum_{cyc}a^2b^2\Leftrightarrow$$

$$\sum_{cyc}a^2b^2\,(a+b)^2+2abc\left(\sum_{cyc}a(a+b)(a+c)\right)$$

$$\geq 4\left(\sum_{cyc}a^3b^3+abc\sum_{cyc}ab(a+b)\right)$$

Which is true because:

$$\sum_{cyc}a^2b^2\,(a+b)^2-4\sum_{cyc}a^3b^3=\sum_{cyc}a^2b^2\,(a-b)^2\geq 0$$

$$2abc\left(\sum_{cyc}a(a+b)(a+c)\right)-4abc\sum_{cyc}ab(a+b)=$$

$= 2abc(\sum_{cyc}a^3+3abc-\sum_{cyc}ab(a+b))\geq 0$ *true by Schur's inequality.*

Proved.Equality for $a=b=c$.

6.117

$$\mathbf{If}\ \mathbf{a_1, a_2, \dots, a_n > 0}, \boldsymbol{then}:$$

$$\prod_{i=1}^{n}\left(1+a_i^{1+a_i^{1+a_i}}\right) \geq 2^n\left(\prod_{i=1}^{n} a_i\right)^{\frac{1}{n}\sum_{i=1}^{n} a_i}$$

Florică Anastase

Solution(Tran Hong)

We have inequality:

$$1+\alpha^{1+\alpha^{1+\alpha}} \geq 2\alpha^{\alpha}, \forall \alpha > 0$$

$$\prod_{i=1}^{n}\left(1+a_i^{1+a_i^{1+a_i}}\right) \geq \prod_{i=1}^{n}\left(2a_i^{a_i}\right)$$

$$\geq 2^n \prod_{i=1}^{n} a_i^{a_i} \overset{Am-Gm}{\geq} 2^n\left(\frac{a_1+a_2+\dots+a_n}{n}\right)^{a_1+a_2+\dots+a_n}$$

$$\overset{Am-Gm}{\geq} 2^n\left(\sqrt[n]{a_1\cdot a_2\cdot\ldots\cdot a_n}\right)^{a_1+a_2+\dots+a_n} = 2^n(a_1\cdot a_2\cdot\ldots\cdot a_n)^{\frac{a_1+a_2+\dots+a_n}{n}} = RHS$$

6.118 If $a, b, c > 0$ then:

$$\frac{a^3+b^3+c^3}{abc}+36\left(\frac{a}{a+b}+\frac{b}{b+c}+\frac{c}{c+a}\right) \geq 57$$

Rahim Shahbazov

Solution (Tran Hong)

$$\frac{a}{a+b}+\frac{b}{b+c}+\frac{c}{c+a} = \frac{a^2}{a^2+ab}+\frac{b^2}{b^2+bc}+\frac{c^2}{c^2+ca} \overset{CBS}{\geq} \frac{(a+b+c)^2}{a^2+b^2+c^2+ab+bc+ca}$$

$$\frac{a^3+b^3+c^3}{abc}+36\left(\frac{a}{a+b}+\frac{b}{b+c}+\frac{c}{c+a}\right) \geq \frac{a^3+b^3+c^3}{abc}+36\cdot\frac{(a+b+c)^2}{a^2+b^2+c^2+ab+bc+ca} \overset{(1)}{\geq} 57$$

Let: $p = a+b+c; q = ab+bc+ca; r = abc$

$$(1) \Leftrightarrow \frac{p^3-3pq+3r}{r}+36\cdot\frac{p^2}{p^2-2q+q} \geq 57$$

$$\frac{p(p^2-3q)}{r}+36\cdot\frac{p^2}{p^2-q}\geq 54$$

But: $(a+b+c)(ab+bc+ca)\geq 9abc\Rightarrow pq\geq 9r\Rightarrow r\leq\frac{pq}{9}$

$$\frac{9p(p^2-3q)}{pq}+36\cdot\frac{p^2}{p^2-q}\overset{(2)}{\geq} 54$$

$$(2)\Leftrightarrow\frac{9(p^2-3q)}{q}+36\left(1+\frac{q}{p^2-q}\right)\geq 54$$

$$\frac{9(p^2-3q)}{q}+\frac{36q}{p^2-q}\geq 18\Leftrightarrow\frac{p^2-3q}{q}+\frac{4q}{p^2-q}\geq 2$$

$$\frac{p^2}{q}+\frac{4q}{p^2-q}\geq 5\Leftrightarrow p^2(p^2-q)+4q^2\geq 5q(p^2-q)$$

$\Leftrightarrow (p^2-3q)^2\geq 0\Rightarrow (2)$ *is true*$\Rightarrow (1)$ *is true.Proved.*

Equality$\Leftrightarrow a=b=c$.

6.119 If $x,y,z>0$ then:

$$\sqrt{\frac{x(x+z)}{y(y+z)}}+\sqrt{\frac{y(y+x)}{z(z+x)}}+\sqrt{\frac{z(z+y)}{x(x+y)}}+\frac{8xyz}{(x+y)(y+z)(z+x)}\geq 4$$

Rahim Shahbazov

Solution (Tran Hong)

Let: $a=\sqrt{\frac{x}{y+z}},b=\sqrt{\frac{y}{z+x}},c=\sqrt{\frac{z}{x+y}}$

$$a^2=\frac{x}{y+z},b^2=\frac{y}{z+x},c^2=\frac{z}{x+y}$$

$$\frac{1}{1+a^2}+\frac{1}{1+b^2}+\frac{1}{1+c^2}=2\Rightarrow 2a^2b^2c^2+a^2b^2+b^2c^2+c^2a^2=1$$

$\Rightarrow\exists\Delta XYZ$ *acute such that:*

$$ab=cosZ;bc=cosX;ca=cosY$$

Hence

$$\sqrt{\frac{x(x+z)}{y(y+z)}}+\sqrt{\frac{y(y+x)}{z(z+x)}}+\sqrt{\frac{z(z+y)}{x(x+y)}}+\frac{8xyz}{(x+y)(y+z)(z+x)}\geq 4$$

$$\frac{a}{b}+\frac{b}{c}+\frac{c}{a}+8(abc)^2\geq 4\Leftrightarrow$$

$$\frac{cosY}{cosX}+\frac{cosZ}{cosY}+\frac{cosX}{cosZ}+8cosXcosYcosZ\overbrace{\geq}^{(1)}4$$

$$\frac{cosY}{cosX}+\frac{cosZ}{cosY}+\frac{cosX}{cosZ}\geq\frac{9(cos^2X+cos^2Y+cos^2Z)}{(cosX+cosY+cosZ)^2}$$

$$\overbrace{\geq}^{0<cosX+cosY+cosZ\leq\frac{3}{2}}\frac{9(cos^2X+cos^2Y+cos^2Z)}{\left(\frac{3}{2}\right)^2}$$

$$=4(cos^2X+cos^2Y+cos^2Z)=4(1-2cosXcosYcosZ)$$

$\Rightarrow (*)$ *is true.Proved.*

6.120 If $a,b,c>0$ prove:

$$\frac{(a^2+a+1)^{\sqrt{3}}(b^2+b+1)^{\sqrt{3}}(c^2+c+1)^{\sqrt{3}}}{e^{2a}\cdot e^{2b}\cdot e^{2c}}\leq 1$$

Daniel Sitaru

Solution(Tran Hong)

First, we prove:

$$\therefore\ 2e^{2x}\geq 2+4x+4x^2\quad (1)$$

$$\therefore \sqrt{3}(2x+1)(x^2+x+1)^{\sqrt{3}-1}\leq\left(3+\sqrt{3}\right)x+\sqrt{3}\quad (2)$$

Proof:

Let: $\varphi(x)=\left(3+\sqrt{3}\right)x+\sqrt{3}-\sqrt{3}(2x+1)(x^2+x+1)^{\sqrt{3}-1}, x\geq 0$

$$\varphi'(x)=2\sqrt{3}(x^2+x+1)^{\sqrt{3}-1}+\sqrt{3}\left(\sqrt{3}-1\right)(2x+1)^2(x^2+x+1)^{\sqrt{3}-2}$$

$$-\sqrt{3}-3\overbrace{\geq}^{x\geq 0}0$$

$$\text{Then: } \varphi(x) \nearrow \text{ on } [0,\infty) \Rightarrow \varphi(x) \geq \varphi(0) = 0 \Rightarrow \text{(2) true}$$

$$\text{Let: } g(x) = 2(2e^{2x} - 2x^2 - 2x - 1), x \geq 0$$

$$g'(x) = 2(2e^{2x} - 4x - 2) = 4(e^{2x} - 2x - 1) \geq 0$$

$$\text{Then: } g(x) \nearrow \text{ on } [0,\infty) \Rightarrow g(x) \geq g(0) = 0 \Rightarrow (1) true$$

$$\text{Now, we must show that: } e^{2x} \geq (x^2 + x + 1)^{\sqrt{3}}, \forall x \geq 0 \quad (3)$$

$$\text{Let: } \varphi(x) = e^{2x} - (x^2 + x + 1)^{\sqrt{3}}, \forall x \geq 0$$

$$\varphi'(x) = 2e^{2x} - \sqrt{3}(2x + 1)\,(x^2 + x + 1)^{\sqrt{3}-1}$$

$$\geq 4x^2\left(1 - \sqrt{3}\right)x + 2 - \sqrt{3} > 0, \forall x \geq 0$$

$$\varphi(x) \geq \varphi(0) = 0 \Rightarrow \text{(3) true}$$

$$\text{Hence: } (a^2 + a + 1)^{\sqrt{3}} \leq e^{2a}$$

$$\prod_{cyc}(a^2 + a + 1)^{\sqrt{3}} \leq \prod_{cyc} e^{2a}$$

$$\text{Equality for } a = b = c = 0$$

6.121 In ΔABC the following relationship holds:

$$\frac{(3a+b+c)^2}{2a+b+2c} + \frac{(a+3b+c)^2}{2a+2b+c} + \frac{(a+b+3c)^2}{a+2b+2c} \leq \frac{19\cdot(a^2+b^2+c^2) + 26\cdot(ab+bc+ca)}{3\cdot(a+b+c)}$$

George Florin Şerban

Solution (Rahim Shahbazov)

$$\begin{cases} 2a + b + c = x \\ a + 2b + c = y \\ a + b + 2c = z \end{cases} \Rightarrow \begin{cases} b = \dfrac{2y + 2z - 3x}{5} \\ c = \dfrac{2x + 2z - 3y}{5} \\ a = \dfrac{2x + 2y - 3z}{5} \end{cases}$$

Inequality becomes:

$$\frac{(x+y-z)^2}{x}+\frac{(y+z-x)^2}{y}+\frac{(x+z-y)^2}{z}$$
$$\leq\frac{23(x^2+y^2+z^2)-14(xy+yz+zx)}{3(x+y+z)}$$
$$x+2(y-z)+\frac{(y-z)^2}{x}+y+2(z-x)+\frac{(z-x)^2}{y}+z+2(x-y)$$
$$+\frac{(x-y)^2}{z}\leq\frac{23(x^2+y^2+z^2)-14(xy+yz+zx)}{3(x+y+z)}$$
$$\frac{(y-z)^2}{x}+\frac{(z-x)^2}{y}+\frac{(x-y)^2}{z}\leq\frac{20(x^2+y^2+z^2-xy-yz-zx)}{3(x+y+z)}$$
$$\sum_{cyc}\frac{(y-z)^2}{x}\leq\sum_{cyc}\frac{10(y-z)^2}{3(x+y+z)}$$
$$\sum_{cyc}(y-z)^2\left[\frac{10}{3(x+y+z)}-\frac{1}{x}\right]\geq 0$$
$$\sum_{cyc}(y-z)^2\frac{7x-3y-3z}{3x(x+y+z)}\geq 0$$
$$7x-3y-3z=5(a+c-b)>0$$

6.122 If $a,b,c>0$ then:

$$\frac{(3a+2b+c+6)(3b+2c+a+6)(3c+2a+b+6)}{(a+1)(b+1)(c+1)}\geq 216$$

Daniel Sitaru

Solution(George Florin Şerban)

Denote: $a+1=x; b+1=y; c+1=z; x,y,z>0\Rightarrow$

$$\frac{(3x+2y+z)(x+3y+2z)(2x+y+3z)}{xyz}\geq 216$$
$$3x+2y+z=x+x+x+y+y+z\overset{Am-Gm}{\geq}6\cdot\sqrt[6]{x^3y^2z}$$
$$x+3y+2z=x+y+y+y+z+z\overset{Am-Gm}{\geq}6\cdot\sqrt[6]{xy^3z^2}$$

$$2x+y+3z=x+x+y+z+z+z \overset{Am-Gm}{\geq} 6\cdot\sqrt[6]{x^2yz^3}$$

$$\prod_{cyc}(3x+2y+z)\geq 6^3\cdot\sqrt[6]{x^6y^6z^6}=216xyz$$

$$\frac{\prod(3x+2y+z)}{xyz}\geq 216$$

Equality for $a=b=c$

6.123 If $x,y,z>0$ then:

$$(\boldsymbol{x}+\boldsymbol{y}+\boldsymbol{z})\left(\frac{\mathbf{1}}{\boldsymbol{x}}+\frac{\mathbf{1}}{\boldsymbol{y}}+\frac{\mathbf{1}}{\boldsymbol{z}}\right)\geq \mathbf{9}\cdot\sqrt{\frac{\boldsymbol{x}^2+\boldsymbol{y}^2+\boldsymbol{z}^2}{\boldsymbol{xy}+\boldsymbol{yz}+\boldsymbol{zx}}}$$

Rahim Shahbazov

Solution(Tran Hong)

$$(x+y+z)\left(\frac{1}{x}+\frac{1}{y}+\frac{1}{z}\right)\geq 9\cdot\sqrt{\frac{x^2+y^2+z^2}{xy+yz+zx}}\quad (*)$$

$$(x+y+z)\left(\frac{1}{x}+\frac{1}{y}+\frac{1}{z}\right)-9\geq 9\cdot\sqrt{\frac{x^2+y^2+z^2}{xy+yz+zx}}-9$$

$$\frac{(y-z)^2}{yz}+\frac{(z-x)^2}{zx}+\frac{(x-y)^2}{xy}\geq\frac{9[(x-y)^2+(y-z)^2+(z-x)^2]}{2\cdot\sqrt{xy+yz+zx}\left(\sqrt{x^2+y^2+z^2}+\sqrt{xy+yz+zx}\right)}$$

$$\sum_{cyc}\left(\frac{1}{xy}-\frac{9}{2\cdot\sqrt{xy+yz+zx}\left(\sqrt{x^2+y^2+z^2}+\sqrt{xy+yz+zx}\right)}\right)(x-y)^2\geq 0$$

$$\sum_{cyc}\frac{\left(2\cdot\sqrt{xy+yz+zx}\left(\sqrt{x^2+y^2+z^2}+\sqrt{xy+yz+zx}\right)-9xy\right)}{xy\cdot 2\cdot\sqrt{xy+yz+zx}\left(\sqrt{x^2+y^2+z^2}+\sqrt{xy+yz+zx}\right)}\cdot(x-y)^2\geq 0$$

$$\sum_{cyc}\left[2\cdot\sqrt{xy+yz+zx}\left(\sqrt{x^2+y^2+z^2}+\sqrt{xy+yz+zx}\right)-9xy\right]\cdot(x-y)^2\geq 0$$

Let: $S=\sum_{cyc}\left[2\cdot\sqrt{xy+yz+zx}\left(\sqrt{x^2+y^2+z^2}+\sqrt{xy+yz+zx}\right)-9xy\right]\cdot$

$$(x-y)^2$$

$$S_z = 2\cdot\sqrt{xy+yz+zx}\left(\sqrt{x^2+y^2+z^2}+\sqrt{xy+yz+zx}\right)-9xy$$

$$S_y = 2\cdot\sqrt{xy+yz+zx}\left(\sqrt{x^2+y^2+z^2}+\sqrt{xy+yz+zx}\right)-9zx$$

$$S_x = 2\cdot\sqrt{xy+yz+zx}\left(\sqrt{x^2+y^2+z^2}+\sqrt{xy+yz+zx}\right)-9yz$$

Now, we have: $\sqrt{x^2+y^2+z^2} \ge \sqrt{xy+yz+zx}$

$$S_x+S_y+S_z \ge 12(xy+yz+zx)-9(xy+yz+zx) = 3(xy+yz+zx) \ge 0; (*)$$

$$S_xS_y+S_yS_z+S_zS_x = 12t^2-36(xy+yz+zx)+81(x^2y^2+y^2z^2+z^2x^2) = \varphi(t),$$

$$t = \sqrt{xy+yz+zx}\left(\sqrt{x^2+y^2+z^2}+\sqrt{xy+yz+zx}\right) > 0$$

$$\varphi'(t) = 0 \Rightarrow t_0 = \frac{3}{2}(xy+yz+zx) \Rightarrow \varphi(t) \ge \varphi(t_0)$$

$$12\cdot\frac{3}{4}(xy+yz+zx)^2-36\cdot\frac{3}{2}(xy+yz+zx)^2+81(x^2y^2+y^2z^2+z^2x^2)$$

$$= 27(xy+yz+zx)^2-54(xy+yz+zx)^2+81(x^2y^2+y^2z^2+z^2x^2)$$

$$\ge 27(xy+yz+zx)^2-54(xy+yz+zx)^2+27(xy+yz+zx)^2 = 0 \Rightarrow$$

$$S_xS_y+S_yS_z+S_zS_x \ge 0; \quad (**)$$

From $(*),(**) \Rightarrow S \ge 0$*.Proved.*

Equality$\Leftrightarrow x = y = z$.

6.124 If $x,y,z > 0, x+y+z = 1, n \ge 2$ then:

$$\frac{x}{\sqrt{nx+y}}+\frac{y}{\sqrt{ny+z}}+\frac{z}{\sqrt{nz+x}} \le \sqrt{\frac{3}{n+1}}$$

Marin Chirciu

Solution(Tran Hong)

$$LHS=\sum_{cyc}\frac{x}{\sqrt{nx+y}} = \sum_{cyc}\sqrt{x}\left(\sqrt{\frac{x}{nx+y}}\right) \overset{BCS}{\le} \sqrt{x+y+z}\cdot\sqrt{\frac{x}{nx+y}+\frac{y}{ny+z}+\frac{z}{nz+x}}$$

$$= \sqrt{\frac{x}{nx+y}+\frac{y}{ny+z}+\frac{z}{nz+x}}$$

$$Put\ \Phi = \frac{x}{nx+y} + \frac{y}{ny+z} + \frac{z}{nz+x} \le \frac{3}{n+1} \Leftrightarrow$$

$$(n+1)[x(ny+z)(nz+x) + y(nx+y)(nz+x) + z(nx+y)(ny+z)]$$
$$\le 3(nx+y)(ny+z)(nz+x) \Leftrightarrow$$
$$(n^2-2n)(x^2y+y^2z+z^2x) + (2n-1)(xy^2+yz^2+zx^2)$$
$$\ge (3n^2-1)xyz; (*)$$

$$Because: n \ge 2 \Rightarrow n^2-2n \ge 0, 2n-1 \ge 3$$

$$(n^2-2n)(x^2y+y^2z+z^2x) \overset{Am-Gm}{\overbrace{\ge}} 3(n^2-2n)\sqrt[3]{(xyz)^3}$$
$$= 3(n^2-2n)xyz; (1)$$

$$(2n-1)(xy^2+yz^2+zx^2) \overset{Am-Gm}{\overbrace{\ge}} 3(2n-1)\sqrt[3]{(xyz)^3} = (2n-1)xyz; (2)$$

$$\overset{(1)+(2)}{\Longrightarrow} (n^2-2n)(x^2y+y^2z+z^2x) + (2n-1)(xy^2+yz^2+zx^2) \ge$$
$$3(n^2-2n)xyz \Rightarrow (*)\ true.$$

$$\Rightarrow \Phi \le \frac{3}{n+1} \Rightarrow LHS \le \sqrt{\Phi} \le \sqrt{\frac{3}{n+1}}. Proved.$$

$$Equality \Leftrightarrow x = y = z = 1$$

6.125 If $a, b, c > 0$ then:

$$\left(\frac{a}{a+b} + \frac{b}{b+c} + \frac{c}{c+a}\right)\sqrt{\frac{a^2+b^2+c^2}{ab+bc+ca}} \ge \frac{3}{2}$$

Rahim Shahbazov

Solution (Soumitra Mandal)

$$\sqrt{\frac{a^2+b^2+c^2}{ab+bc+ca}}\sum_{cyc}\frac{a}{a+b} = \sqrt{\frac{a^2+b^2+c^2}{ab+bc+ca}}\sum_{cyc}\frac{a^2}{a^2+ab}$$

$$\overset{Bergstrom}{\overbrace{\ge}} \sqrt{\frac{a^2+b^2+c^2}{ab+bc+ca}} \cdot \frac{(a+b+c)^2}{a^2+b^2+c^2+ab+bc+ca} = \frac{x+2y}{x+y}\sqrt{\frac{x}{y}}$$

Where $x = a^2+b^2+c^2, y = ab+bc+ca$ *and* $x \ge y$

We need to prove: $\frac{x+2y}{x+y}\sqrt{\frac{x}{y}} \geq \frac{3}{2} \Leftrightarrow 4x(x+2y)^2 \geq 9y(x+y)^2 \Leftrightarrow$

$4x(x^2+4xy+4y^2) \geq 9y(x^2+2xy+y^2) \Leftrightarrow 4x^3+7x^2y-2xy^2-9y^3 \geq 0 \Leftrightarrow$

$(x-y)(4x^2+11xy+9y^2) \geq 0$, *which is true* $x-y \geq 0$

$\sum_{cyc}\frac{a}{a+b}\cdot\sqrt{\frac{a^2+b^2+c^2}{ab+bc+ca}} \geq \frac{3}{2}$.*Proved.*

6.126 If $x_1, x_2, \dots, x_n > 0$ then:

$$2\sum_{i=1}^{n}\left(\frac{1}{x_i+x_{i+1}}\prod_{\substack{j=1\\ j\neq i}}^{n} x_j\right) \leq \prod_{\substack{j=1\\ j\neq i}}^{n} x_j, \quad x_{n+1}=x_1$$

Marin Chirciu

Solution (Tran Hong)

$$2\sum_{i=1}^{n}\left(\frac{1}{x_i+x_{i+1}}\prod_{\substack{j=1\\ j\neq i}}^{n} x_j\right) = 2\left(\frac{x_2x_3\dots x_n}{x_1+x_2}+\frac{x_1x_3\dots x_n}{x_2+x_3}+\dots+\frac{x_1x_2\dots x_{n-1}}{x_n+x_1}\right)$$

$$= 2x_1x_2\dots x_n\left(\frac{1}{x_1(x_1+x_2)}+\frac{1}{x_2(x_2+x_3)}+\frac{1}{x_n(x_n+x_1)}\right) = \Omega$$

$$\frac{2}{2x_1(x_1+x_2)} \overset{CBS}{\leq} 2\cdot\frac{1}{4}\cdot\left(\frac{1}{2x_1}+\frac{1}{x_1+x_2}\right) \leq \frac{2}{4}\cdot\left(\frac{1}{2x_1}+\frac{1}{4}\left(\frac{1}{x_1+x_2}\right)\right)$$

$$= 2\left(\frac{3}{16x_1}+\frac{1}{16x_2}\right)$$

and analogs.

$$\Rightarrow \Omega \leq 4x_1x_2\dots x_n\left[\frac{1}{16}\left(\left(\frac{3}{x_1}+\frac{1}{x_2}\right)+\left(\frac{3}{x_2}+\frac{1}{x_3}\right)+\dots+\left(\frac{3}{x_n}+\frac{1}{x_1}\right)\right)\right]$$

$$= 4x_1x_2 \dots x_n\left(\frac{1}{16}\sum_{i=1}^{n}\frac{4}{x_i}\right) = 4\cdot\frac{4}{16}x_1x_2 \dots x_n\sum_{i=1}^{n}\frac{1}{x_i} = \sum_{i=1}^{n}\left(\prod_{\substack{j=1\\ i\neq j}}^{n}x_i\right)$$

6.127 If $x, y, z, t >, xyzt = 1$ then:

$$\frac{x^2+1}{x^5+3}+\frac{y^2+1}{y^5+3}+\frac{z^2+1}{z^5+3}+\frac{t^2+1}{t^5+3}\leq 2$$

Rahim Shahbazov

Solution(Tran Hong)

$$\frac{x^2+1}{x^5+3}\leq\frac{2}{x+3}\quad (*)$$

$$\Leftrightarrow 2(x^5+3)-(x+3)(x^2+1)\geq 0$$

$$2x^5-x^3-3x^2-x+3\geq 0$$

$$(x-1)^2(x+1)(2x^2+2x+3)\geq 0 \text{ true for } x\geq 0 \Rightarrow (*) \text{ is true.}$$

$$LHS=\sum_{cyc}\frac{x^2+1}{x^5+3}\leq\sum_{cyc}\frac{2}{x+3}=2\sum_{cyc}\frac{1}{x+3}=2\left(\frac{1}{x+3}+\frac{1}{y+3}+\frac{1}{z+3}+\frac{1}{t+3}\right)=2\Phi$$

$$\Phi=\frac{1}{x+3}+\frac{1}{y+3}+\frac{1}{z+3}+\frac{1}{t+3}\leq 1\Leftrightarrow$$

$$\sum_{cyc}(x+3)(y+3)(z+3)\leq(x+3)(y+3)(z+3)(t+3)\Leftrightarrow$$

$$xyzt+2(xyz+xyt+xzt+yzt)+3(xy+xz+xt+yz+yt+zt)\geq 27\overset{xyzt=1}{\Longleftrightarrow}$$

$$2(xyz+xyt+xzt+yzt)+3(xy+xz+xt+yz+yt+zt)\geq 26\quad (**)$$

$$2(xyz+xyt+xzt+yzt)\overset{Am-Gm}{\geq}2\cdot 4\sqrt[4]{(xyzt)^3}=8\quad (1)$$

$$3(xy+xz+xt+yz+yt+zt)\overset{Am-Gm}{\geq}3\cdot 6\sqrt[6]{(xyzt)^3}=18\quad (2)$$

$\overset{(1)+(2)}{\Longrightarrow}(**)$ *is true.Proved. Equality*$\Leftrightarrow x=y=z=t=1$

6.128 If $a, b, c > 0, abc = 1$ then:

$$\frac{a^2+b^2+c^2}{ab+bc+ca}+\frac{3(a+b+c)}{(ab+bc+ca)^2}\geq 2$$

George Apostolopoulos

Solution(Adrian Popa)

$$\frac{a^2+b^2+c^2}{ab+bc+ca}+\frac{3(a+b+c)}{(ab+bc+ca)^2}\geq 2\Leftrightarrow$$

$$(a^2+b^2+c^2)(ab+bc+ca)+3abc\geq 2(ab+bc+ca)^2\Leftrightarrow$$

$$a^3b+b^3c+c^3a+a^3c+b^3a+c^3b\geq 2(a^2b^2+b^2c^2+c^2a^2)\Leftrightarrow$$

$$\frac{a^2}{c}+\frac{b^2}{a}+\frac{c^2}{b}+\frac{a^2}{b}+\frac{b^2}{c}+\frac{c^2}{a}\geq\frac{2}{c^2}+\frac{2}{b^2}+\frac{2}{a^2}$$

$$\frac{a^2+b^2}{c}+\frac{b^2+c^2}{a}+\frac{a^2+c^2}{b}\overset{Am-Gm}{\geq}\frac{2ab}{c}+\frac{2bc}{a}+\frac{2ca}{b}=\frac{2}{c^2}+\frac{2}{b^2}+\frac{2}{a^2}$$

6.129 If $a, b, c > 0, abc = 1$ then:

$$\frac{a^2+b^2+c^2}{ab+bc+ca}+\frac{3(a+b+c)}{(ab+bc+ca)^2}\geq 2$$

George Apostolopoulos

Solution(Adrian Popa)

$$\frac{a^2+b^2+c^2}{ab+bc+ca}+\frac{3(a+b+c)}{(ab+bc+ca)^2}\geq 2\Leftrightarrow$$

$$(a^2+b^2+c^2)(ab+bc+ca)+3abc\geq 2(ab+bc+ca)^2\Leftrightarrow$$

$$a^3b+b^3c+c^3a+a^3c+b^3a+c^3b\geq 2(a^2b^2+b^2c^2+c^2a^2)\Leftrightarrow$$

$$\frac{a^2}{c}+\frac{b^2}{a}+\frac{c^2}{b}+\frac{a^2}{b}+\frac{b^2}{c}+\frac{c^2}{a}\geq\frac{2}{c^2}+\frac{2}{b^2}+\frac{2}{a^2}$$

$$\frac{a^2+b^2}{c}+\frac{b^2+c^2}{a}+\frac{a^2+c^2}{b}\overset{Am-Gm}{\geq}\frac{2ab}{c}+\frac{2bc}{a}+\frac{2ca}{b}=\frac{2}{c^2}+\frac{2}{b^2}+\frac{2}{a^2}$$

6.130 If $a, b, c > 0$ then:

$$\left(\frac{1}{a}+\frac{1}{b}+\frac{1}{c}\right)(9+a^2b^2+b^2c^2+c^2a^2) \geq 36$$

Nguyen Van Canh

Solution(Marian Ursărescu)

From $x^2+y^2+z^2 \geq xy+yz+zx$ *we have:*

$$a^2b^2+b^2c^2+c^2a^2 \geq abc(a+b+c) \geq 3abc\sqrt[3]{abc} \Rightarrow$$

$$a^2b^2+b^2c^2+c^2a^2 \geq 3\sqrt[3]{a^4b^4c^4}$$

We must show that: $\left(\frac{1}{a}+\frac{1}{b}+\frac{1}{c}\right)(9+3\sqrt[3]{a^4b^4c^4}) \geq 36$

$$(ab+bc+ca)\left(3+\sqrt[3]{a^4b^4c^4}\right) \geq 12abc \quad (1)$$

But: $ab+bc+ca \geq 3\sqrt[3]{a^2b^2c^2} \quad (2)$

$$3+\sqrt[3]{a^4b^4c^4} = 1+1+1+\sqrt[3]{a^4b^4c^4} \geq 4\sqrt[4]{\sqrt[3]{a^4b^4c^4}} = 4\sqrt[3]{abc} \quad (3)$$

From (1)+(2) we have:

$$(ab+bc+ca)\left(3+\sqrt[3]{a^4b^4c^4}\right) \geq 12\sqrt[3]{a^3b^3c^3} = 12abc$$

6.131 Let $x, y, z > 0$. Prove:

$$x^3+y^3+z^3-(x+y+z) \geq 2log(xyz)$$

Jalil Hajimir

Solution (Daniel Sitaru)

$$f:(0,\infty) \to \mathbb{R}, f(x) = x^3-x-2logx$$

$$f'(x) = 3x^2-1-\frac{2}{x} = \frac{(x-1)(3x^2+3x+2)}{x}$$

$$sgnf'(x) = sgn(x-1), min\, f(x) = f(1) = 0 \to f(x) \geq 0$$

$$\begin{cases} x^3-x-2logx \geq 0 \\ y^3-y-2logy \geq 0 \\ z^3-z-2logz \geq 0 \end{cases} \to \begin{cases} x^3-x \geq 2logx \\ y^3-y \geq 2logy \\ z^3-z \geq 2logz \end{cases} \to$$

$$\to x^3+y^3+z^3-(x+y+z) \geq 2logx+2logy+2logz = 2log(xyz)$$

Equality holds for $x = y = z = 1$.

6.132 If $\boldsymbol{a, b, c, n > 0}$,

$\boldsymbol{(a^2 - na + n^2)(b^2 - nb + n^2)(c^2 - nc + n^2) = 1}$ **then:**

$$\boldsymbol{a^2b^2 + b^2c^2 + c^2a^2 + n^4 \leq \frac{4}{n^2}}$$

Marin Chirciu, Octavian Stroe

Solution (Şerban Florin George)

$$(x-1)^4 \geq 0, \forall x \in \mathbb{R} \Leftrightarrow x^4 - 4x^3 + 6x^2 - 4x + 1 \geq 0 \Leftrightarrow$$

$$(x^4 - 2x^3 + 3x^2 - 2x + 1) + (-2x^3 + 3x^2 - 2x) \geq 0$$

$$(x^2 - x + 1)^2 + (x^4 - 2x^3 + 3x^2 - 2x + 1) \geq x^4 + 1$$

$$(x^2 - x + 1)^2 \geq x^4 + 1 \Rightarrow \prod_{cyc}(a^2 - na + n^2) = 1 \Rightarrow \prod_{cyc}\left(\left(\frac{a}{n}\right)^2 - \frac{a}{n} + 1\right)$$

$$= \frac{1}{n^6}$$

Denote: $\frac{a}{n} = x; \frac{b}{n} = y; \frac{c}{z} = z \Rightarrow \prod_{cyc}(x^2 - x + 1)^2 = \frac{1}{n^{12}}$

$$\prod_{cyc}(x^2 - x + 1)^2 = \frac{1}{n^{12}} \geq \frac{(x^4+1)(y^4+1)(z^4+1)}{2 \cdot 2 \cdot 2}$$

$$= \frac{(2x^4+2)(x^4y^4 + x^4 + y^4 + 1)}{16}$$

$$= \frac{(x^4 + x^4 + 1 + 1)(x^4y^4 + x^4 + y^4 + 1)}{16}$$

$$= \frac{((x^2)^2 + (x^2)^2 + 1^2 + 1^2)((x^2)^2 + (y^2)^2 + (x^2)^2(y^2)^2 + 1^2}{16}$$

$$\overset{C.B.S}{\geq} \frac{(x^2 + y^2 + x^2y^2 + 1)^2}{16} \Rightarrow \frac{1}{n^6} \geq \frac{\sum_{cyc} x^2y^2 + 1}{4}$$

$$\frac{4}{n^6} \geq \sum_{cyc}\frac{a^2b^2}{n^4} + 1 \Rightarrow \frac{4}{n^2} \geq \sum_{cyc} a^2b^2 + n^4$$

6.133 If $x, y, z \geq 0, x + y + z = 1$ then:

$$\mu(x^3 + y^3 + z^3) + 1 \geq \left(3 + \frac{\mu}{3}\right)(x^2 + y^2 + z^2), \mu \geq 6$$

Marin Chirciu

Solution (Tran Hong)

$$\textit{Because: } x + y + z = 1$$

$$\mu(x^3 + y^3 + z^3) + 1 \geq \left(3 + \frac{\mu}{3}\right)(x^2 + y^2 + z^2) \Leftrightarrow$$

$$3\mu(x^3 + y^3 + z^3) + 3(x + y + z)^3 \overset{(*)}{\geq} (9 + \mu)(x^2 + y^2 + z^2)(x + y + z)$$

$$\textit{Let: } p = x + y + z; q = xy + yz + zx; r = xyz$$

$$(*) \Leftrightarrow 3\mu(p^3 - 3pq + 3r) + 3p^3 \geq (9 + \mu)(p^2 - 2q)p \Leftrightarrow$$

$$2(\mu - 3)p^3 + (18 - 7\mu)pq + 9\mu r \geq 0$$

$$r \overset{Schur's}{\geq} \frac{p(4q - p^2)}{9} \Rightarrow 9r \geq 4pq - p^3 \Rightarrow 9\mu r \geq 4\mu pq - \mu p^3$$

We need to prove:

$$2(\mu - 3)p^3 + (18 - 7\mu) + 4\mu pq - \mu p^3 \geq 0 \Leftrightarrow (\mu - 6)p^3 + 3(6 - \mu)pq \geq 0$$

$$\Leftrightarrow (\mu - 6) \cdot p \cdot (p^2 - 3q) \geq 0$$

$$\textit{Which is true because: } \mu \geq 6 \Rightarrow \mu - 6 \geq 0$$

$$(a + b + c)^2 \geq 3(ab + bc + ca) \Rightarrow p^2 \geq 3q.$$

6.134 If $x, y, z \geq 2$ then:

$$\sum_{cyc} \frac{1}{x + 1} = 1 \Rightarrow \sum_{cyc} \frac{3x^2 + x + 4}{(x + 1)(x^4 + 2)} + 2 \leq 2\left(\frac{1}{x} + \frac{1}{y} + \frac{1}{z}\right)$$

Daniel Sitaru

Solution by (Tran Hong)

For $x, y, z \geq 2$ *and* $\frac{1}{x+1} + \frac{1}{y+1} + \frac{1}{z+1} = 1$. *We may write Inequality as:*

$$\frac{3x^2+x+4}{(x+1)(x^4+2)}+\frac{3y^2+y+4}{(y+1)(y^4+2)}+\frac{3z^2+z+4}{(z+1)(x^4+2)}$$
$$+2\left(\frac{1}{x+1}+\frac{1}{y+1}+\frac{1}{z+1}\right)\le 2\left(\frac{1}{x}+\frac{1}{y}+\frac{1}{z}\right);$$
$$\leftrightarrow \sum\left(\frac{3x^2+x+4}{(x+1)(x^4+2)}+\frac{2}{x+1}\right)\le 2\left(\frac{1}{x}+\frac{1}{y}+\frac{1}{z}\right);$$
$$\leftrightarrow \sum\left(\frac{2x^4+3x^2+x+8}{(x+1)(x^4+2)}\right)\le \sum\frac{2}{x};\quad (*)$$

Hence, we must show that:

$$\frac{2x^4+3x^2+x+8}{(x+1)(x^4+2)}\le\frac{2}{x};\ (\ \forall x\ge 2\ )$$
$$\leftrightarrow\ x(2x^4+3x^2+x+8)\le 2(x+1)(x^4+2)$$

$$\leftrightarrow 2x^4-3x^3-x^2-4x+4\ge 0\leftrightarrow (x-2)(2x^3+x^2+x-2)\ge 0$$

Which is true because:

$$x\ge 2\ \to x-2\ge 0;\ 2x^3+x^2+x-2\ge 16+4+2-2=20>0$$

Similary:

$$\frac{2y^4+3y^2+y+8}{(y+1)(y^4+2)}\le\frac{2}{y};\ (\ \forall y\ge 2\ )$$

$$\frac{2z^4+3z^2+z+8}{(z+1)(z^4+2)}\le\frac{2}{z};\ (\ \forall z\ge 2\ )\to (*)\ is\ true.\ \ Proved.$$

6.135 If $a,b,c>0$ then:

$$\frac{\left(\sum_{cyc}ab\right)\left(\sum_{cyc}\frac{1}{ab}\right)}{\left(\sum_{cyc}\sqrt[3]{a}\right)\left(\sum_{cyc}\sqrt[3]{a^2}\right)}\ge\frac{\left(\sum_{cyc}\frac{1}{\sqrt[3]{a}}\right)\left(\sum_{cyc}\frac{1}{\sqrt[3]{a^2}}\right)}{\left(\sum_{cyc}a^2b^2\right)\left(\sum_{cyc}\frac{1}{a^2b^2}\right)}$$

Daniel Sitaru

Solution(Sudhir Jha)

$$\frac{\left(\sum_{cyc} ab\right)\left(\sum_{cyc}\frac{1}{ab}\right)}{\left(\sum_{cyc}\sqrt[3]{a}\right)\left(\sum_{cyc}\sqrt[3]{a^2}\right)} \ge \frac{\left(\sum_{cyc}\frac{1}{\sqrt[3]{a}}\right)\left(\sum_{cyc}\frac{1}{\sqrt[3]{a^2}}\right)}{\left(\sum_{cyc} a^2b^2\right)\left(\sum_{cyc}\frac{1}{a^2b^2}\right)};\ \ (1)$$

Hence

$$\left(\sum_{cyc} ab\right)\left(\sum_{cyc}\frac{1}{ab}\right)\left(\sum_{cyc} a^2b^2\right)\left(\sum_{cyc}\frac{1}{a^2b^2}\right) \ge$$
$$\left(\sum_{cyc}\sqrt[3]{a}\right)\left(\sum_{cyc}\sqrt[3]{a^2}\right)\left(\sum_{cyc}\frac{1}{\sqrt[3]{a}}\right)\left(\sum_{cyc}\frac{1}{\sqrt[3]{a^2}}\right)$$

Hence

$$\left(\sum_{cyc} ab\right)\left(\sum_{cyc}\frac{c}{abc}\right)\left(\sum_{cyc} a^2b^2\right)\left(\sum_{cyc}\frac{c^2}{a^2b^2c^2}\right) \ge$$
$$\left(\sum_{cyc}\sqrt[3]{a}\right)\left(\sum_{cyc}\sqrt[3]{a^2}\right)\left(\sum_{cyc}\frac{\sqrt[3]{bc}}{\sqrt[3]{abc}}\right)\left(\sum_{cyc}\frac{\sqrt[3]{b^2c^2}}{\sqrt[3]{a^2b^2c^2}}\right)$$

Hence

$$\left(\sum_{cyc} ab\right)\left(\sum_{cyc} a\right)\left(\sum_{cyc} a^2b^2\right)\left(\sum_{cyc} a^2\right) \ge$$
$$a^2b^2c^2\left(\sum_{cyc}\sqrt[3]{a}\right)\left(\sum_{cyc}\sqrt[3]{a^2}\right)\left(\sum_{cyc}\sqrt[3]{ab}\right)\left(\sum_{cyc}\sqrt[3]{a^2b^2}\right);\ \ (2)$$

By Chebyshev's inequality, we have:

$$\sum_{cyc} ab \ge \frac{\left(\sum_{cyc}\sqrt[3]{ab}\right)\left(\sum_{cyc}\sqrt[3]{a^2b^2}\right)}{3};\ \ (3)$$

$$\left(\sum_{cyc} a\right) \ge \frac{\left(\sum_{cyc}\sqrt[3]{a}\right)\left(\sum_{cyc}\sqrt[3]{a^2}\right)}{3};\ \ (4)$$

$$\left(\sum_{cyc} a^2b^2\right)\left(\sum_{cyc} a^2\right) \overset{Am-Gm}{\ge} 3\cdot\sqrt[3]{\prod_{cyc} a^4}\cdot 3\cdot\sqrt[3]{\prod_{cyc} a^2} = 9a^2b^2c^2;\ \ (5)$$

Multiplying (3),(4),(5) we get (2) is true, then (1) is true.Proved.

6.136 If $a,b,c>2, a+b+c=9$ then:

$$\Gamma\left(\frac{a}{b}\right)+\Gamma\left(\frac{b}{c}\right)+\Gamma\left(\frac{c}{a}\right)\geq 6$$

Jalil Hajimir

Solution (Daniel Sitaru)

$$\Gamma\left(\frac{a}{b}\right)+\Gamma\left(\frac{b}{c}\right)+\Gamma\left(\frac{c}{a}\right)=$$

$$=\sum_{cyc}\left(\frac{2}{3}\Gamma\left(\frac{a}{b}\right)+\frac{1}{3}\Gamma\left(\frac{b}{c}\right)\right)\overset{JENSEN}{\geq}\sum_{cyc}\Gamma\left(\frac{\frac{2a}{b}+\frac{b}{c}}{3}\right)\geq$$

$$\overset{JENSEN}{\geq}3\Gamma\left(\sum_{cyc}\frac{\frac{2a}{b}+\frac{b}{c}}{3}\right)=3\Gamma\left(\frac{\frac{2a}{b}+\frac{b}{c}+\frac{2b}{c}+\frac{c}{a}+\frac{2c}{a}+\frac{a}{b}}{3}\right)=$$

$$=3\Gamma\left(\frac{a}{b}+\frac{b}{c}+\frac{c}{a}\right)\overset{a,b,c>2}{\geq}3\Gamma\left(3\sqrt[3]{\frac{a}{b}\cdot\frac{b}{c}\cdot\frac{c}{a}}\right)=3\Gamma(3)=3\cdot 2!=6$$

Equality holds for $a=b=c=3$.

6.137 If $a,b,c>0$ then:

$$\frac{(a^2-ab+b^2)^6}{(a+b)^{12}}+\frac{(b^2-bc+c^2)^6}{(b+c)^{12}}+\frac{(c^2-ca+a^2)^6}{(c+a)^{12}}\geq\frac{3}{4096}$$

Daniel Sitaru

Solution(Abner Chinga Bazo)

$$(a-b)^2\geq 0\Leftrightarrow a^2+b^2\geq 2ab\Leftrightarrow 3(a^2+b^2)\geq 6ab$$

$$4(a^2+b^2)-4ab\geq a^2+b^2+2ab\Leftrightarrow 4(a^2-ab+b^2)\geq(a+b)^2$$

$$\Leftrightarrow\frac{a^2-ab+b^2}{(a+b)^2}\geq\frac{1}{4}\Leftrightarrow\frac{(a^2-ab+b^2)^6}{(a+b)^{12}}\geq\frac{1}{2^{12}}$$

$$\frac{(a^2-ab+b^2)^6}{(a+b)^{12}}+\frac{(b^2-bc+c^2)^6}{(b+c)^{12}}+\frac{(c^2-ca+a^2)^6}{(c+a)^{12}}\geq\frac{3}{4096}$$

6.138 If $a,b,c>0,\mu\geq 0,a+b+c+\mu abc=8\mu+6$ then:

$$\left(1+\mu ab+\frac{b}{c}\right)\left(1+\mu bc+\frac{c}{a}\right)\left(1+\mu ca+\frac{a}{b}\right)\geq 8(1+2\mu)^3$$

Marin Chirciu, Daniel Văcaru

Solution by (Tran Hong)

$$a,b,c>0,a+b+c+\mu abc=8\mu+6\Rightarrow \mu abc=8\mu+6-(a+b+c)$$

$$\left(1+\mu ab+\frac{b}{c}\right)\left(1+\mu bc+\frac{c}{a}\right)\left(1+\mu ca+\frac{a}{b}\right)\geq 8(1+2\mu)^3$$

$$\Leftrightarrow (b+c+\mu abc)(a+c+\mu abc)(a+b+\mu abc)\geq 8abc(1+2\mu)^3$$

$$\Leftrightarrow (8\mu+6-a)(8\mu+6-b)(8\mu+6-c)\geq 8abc(1+2\mu)^3$$

$$\overset{k=8\mu+6}{\Longleftrightarrow}(k-a)(k-b)(k-c)\geq 8abc(1+2\mu)^3$$

$$\Leftrightarrow k^3-(a+b+c)k^2+(ab+bc+ca)k-abc\geq 8abc(1+2\mu)^3$$

$$\Leftrightarrow k^3-(a+b+c)k^2+(ab+bc+ca)k\geq abc[8(1+2\mu)^3+1]$$

$$\Leftrightarrow k^3+(\mu abc-8\mu-6)k^2+(ab+bc+ca)k\geq abc[8(1+2\mu)^3+1]$$

$$\Leftrightarrow k^3-(8\mu+6)k^2+(ab+bc+ca)k\geq abc[8(1+2\mu)^3+1-\mu k^2]$$

$$\Leftrightarrow (8\mu+6)^3-(8\mu+6)(8\mu+6)^2+(8\mu+6)(ab+bc+ca)$$

$$\geq abc[8(1+2\mu)^3+1-\mu(8\mu+6)^2]$$

$$\Leftrightarrow 2(4\mu+3)(ab+c+a)\geq abc(4\mu+3)$$

$$\overset{4\mu+3>0}{\Longleftrightarrow}2(ab+bc+ca)\overset{(*)}{\geq}3abc;$$

$$8\mu+6=a+b+c+\mu abc\overset{Am-Gm}{\geq}3\sqrt[3]{abc}+\mu abc$$

$$\Leftrightarrow \mu t^3+3t-(8\mu+6)\leq 0\Leftrightarrow \mu(t-2(t^2+2t+)+3(t-2)\leq 0$$

$$\Leftrightarrow (t-2)(\mu t^2+2\mu t+4\mu+3)\leq 0\overset{t>0,\mu\geq 0}{\Longleftrightarrow}t\leq 2\Leftrightarrow abc\leq 2$$

So, $2(ab+bc+ca)\overset{Am-Gm}{\geq}2\cdot 3\sqrt[3]{(abc)^2}\overset{(**)}{\geq}3abc$

$(**)\Leftrightarrow 2\sqrt[3]{(abc)^2}\geq abc\Leftrightarrow 8(abc)^2\geq (abc)^3\Leftrightarrow 2\geq abc$ *(true), then (*) is true.Proved.*

6.139 If $a, b, c, d > 0, abcd = 1$ then:

$$\sum_{cyc} \frac{1+(a^3+b^3+c^3)d}{a+b+c} \geq \frac{4}{3}\left(\frac{1}{a}+\frac{1}{b}+\frac{1}{c}+\frac{1}{d}\right)$$

Marin Chirciu

Solution (Tran Hong)

$$a, b, c, d > 0, abcd = 1 \Rightarrow$$

$$\frac{1+(a^3+b^3+c^3)d}{a+b+c} = \frac{(a^3+b^3+c^3+abc)}{(a+b+c)abc} \overset{(1)}{\geq} \frac{4}{9}\left(\frac{1}{a}+\frac{1}{b}+\frac{1}{c}\right)$$

$$(1) \Leftrightarrow 9(a^3+b^3+c^3+abc) \geq 4(a+b+c)(ab+bc+ca)$$

$$\Leftrightarrow\ 9(a^3+b^3+c^3)+9abc \overset{(*)}{\geq} 4(a+b+c)(ab+bc+ca)$$

By Schur's Inequality:

$$a^3+b^3+c^3+3abc \geq ab(a+b)+bc(c+b)+ca(c+a)$$

$$(a+b+c)^3+9abc \geq 4(a+b+c)(ab+bc+ca)$$

But: $a^3+b^3+c^3 \geq \frac{(a+b+c)^3}{3^2} \Leftrightarrow 9(a^3+b^3+c^3) \geq (a+b+c)^3$

$\Leftrightarrow 9(a^3+b^3+c^3)+9abc \geq (a+b+c)^3+9abc \geq 4(a+b+c)(ab+bc+ca) \Rightarrow (*)$ *is true* $\Rightarrow (1)$ *is true.*

Similarly:

$$\frac{1+(b^3+c^3+d^3)a}{b+c+d} \geq \frac{4}{9}\left(\frac{1}{b}+\frac{1}{c}+\frac{1}{d}\right)$$

$$\frac{1+(c^3+d^3+a^3)d}{c+d+a} \geq \frac{4}{9}\left(\frac{1}{c}+\frac{1}{d}+\frac{1}{a}\right)$$

$$\frac{1+(a^3+b^3+d^3)d}{a+b+d} \geq \frac{4}{9}\left(\frac{1}{a}+\frac{1}{b}+\frac{1}{d}\right)$$

$$\Rightarrow \sum_{cyc} \frac{1+(a^3+b^3+c^3)d}{a+b+c} \geq \frac{4}{9}\left(\frac{3}{a}+\frac{3}{b}+\frac{3}{c}+\frac{3}{d}\right) = \frac{4}{3}\left(\frac{1}{a}+\frac{1}{b}+\frac{1}{c}+\frac{1}{d}\right)$$

6.140 If $a,b,c>0$ then:

$$\frac{4}{ac+b\sqrt{a}}+\frac{4}{ab+c\sqrt{b}}+\frac{4}{bc+a\sqrt{c}}\le\frac{1+\sqrt{a}}{bc}+\frac{1+\sqrt{b}}{ca}+\frac{1+\sqrt{c}}{ab}$$

Florică Anastase

Solution(Tran Hong)

For $x,y>0$ we have:

$$(x+y)\left(\frac{1}{x}+\frac{1}{y}\right)\ge 4\Leftrightarrow\frac{4}{x+y}\le\frac{1}{x}+\frac{1}{y};(*)$$

Using $(*)$ inequality:

$$\frac{4}{ac+b\sqrt{a}}\le\frac{1}{ac}+\frac{1}{b\sqrt{a}}=\frac{1}{ac}+\frac{\sqrt{b}}{ba};(1)$$

$$\frac{4}{ab+c\sqrt{b}}\le\frac{1}{ab}+\frac{1}{c\sqrt{b}}=\frac{1}{ab}+\frac{\sqrt{b}}{bc};(2)$$

$$\frac{4}{bc+a\sqrt{c}}\le\frac{1}{bc}+\frac{1}{a\sqrt{c}}=\frac{1}{bc}+\frac{\sqrt{c}}{ac};(3)$$

From $(1),(2),(3)$ we have:

$$LHS\le\frac{1+\sqrt{a}}{bc}+\frac{1+\sqrt{b}}{ca}+\frac{1+\sqrt{c}}{ab}=RHS$$

6.141 In ΔABC the following relationship holds:

$$\frac{a}{a+\mu(b+c)}+\frac{b}{b+\mu(c+a)}+\frac{c}{c+\mu(a+b)}\ge\frac{3}{1+2\mu},\mu\ge 1$$

Marin Chirciu

Solution(Florică Anastase)

$$\sum_{cyc}\frac{a}{a+\mu(b+c)}=\sum_{cyc}\frac{a^2}{a^2+\mu(ab+ac)}\overset{Bergstrom}{\ge}\frac{(\sum a)^2}{\sum a^2+2\mu\sum ab}\ge\frac{3}{1+2\mu}$$

$$(1+2\mu)\left(\sum a\right)^2\ge 3\sum a^2+6\mu\sum ab$$

$$2(\mu-1)\sum a^2 \geq 2(\mu-1)\sum ab$$

$$(\mu-1)[(a-b)^2+(b-c)^2+(c-a)^2] \geq 0 \text{ true.}$$

Equality holds if $a=b=c \Leftrightarrow \Delta ABC$ is equilateral.

6.142 If $0<a,b,c<1$ then:

$$\prod_{cyc}\frac{(1+ab)(1+ac)}{1+a\sqrt{bc}} \geq \left(1+\sqrt[3]{a^2b^2c^2}\right)^3$$

Florică Anastase

Solution (Tran Hong)

For $x,y,z>0$ we have:

$$(1+x)(1+y) \geq \left(1+\sqrt{xy}\right)^2 \Leftrightarrow 1+x+y+xy \geq 1+2\sqrt{xy}+xy \Leftrightarrow$$

$$x+y-2\sqrt{xy} \geq 0 \Leftrightarrow \left(\sqrt{x}-\sqrt{y}\right)^2 \geq 0 \text{ true.}$$

$$(1+x)(1+y)(1+z) \geq \left(1^3+\sqrt[3]{x^3}\right)\left(1^3+\sqrt[3]{y^3}\right)\left(1^3+\sqrt[3]{z^3}\right) \overset{Holder}{\geq}$$

$$\geq \left(1\cdot 1\cdot 1+\sqrt[3]{xyz}\right)^3 = \left(1+\sqrt[3]{xyz}\right)^3; (*)$$

Now,

$$\frac{(1+ab)(1+ac)}{1+a\sqrt{bc}} \geq \frac{\left(1+\sqrt{ab\cdot ac}\right)^2}{1+a\sqrt{bc}} = 1+a\sqrt{bc}$$

$$\prod_{cyc}\frac{(1+ab)(1+ac)}{1+a\sqrt{bc}} \geq \prod_{cyc}\left(1+a\sqrt{bc}\right) \overset{by(*)}{\geq} \left(1+\sqrt[3]{a^2b^2c^2}\right)^3$$

6.143 If $x,y,z>0, \mu \geq 0$ then:

$$\sum_{cyc}\left(\frac{x^4}{x^3+\mu y^3}\right)^3 \geq \frac{x^3+y^3+z^3}{(1+\mu)^3}$$

Marin Chirciu

Solution(Florică Anastase)

$$\sum_{cyc}\left(\frac{x^4}{x^3+\mu y^3}\right)^3=\sum_{cyc}\frac{(x^3)^4}{(x^3+\mu y^3)^3}\overset{Radon}{\geq}\frac{(x^3+y^3+z^3)^4}{(1+\mu)^3(x^3+y^3+z^3)^3}=$$

$$=\frac{x^3+y^3+z^3}{(1+\mu)^3}$$

6.144 If $\boldsymbol{a,b.c>0}$, $\boldsymbol{abc=1}$ then:

$$\boldsymbol{\frac{a}{a^5+1}+\frac{b}{b^5+1}+\frac{c}{c^5+1}\leq\frac{3}{2}}$$

Jalil Hajimir

Solution (Tran Hong)

First, for all $x>0$ *we need to prove:*

$$\frac{x}{x^5+1}\leq\frac{1}{x^3+1}\leftrightarrow x^5+1\geq x(x^3+1)\leftrightarrow x^5-x^4-x+1\geq 0$$

$$\leftrightarrow (x-1)^2(x^2+1)\geq 0\ (\therefore\ true\ for\ x>0)$$

$$Equality\leftrightarrow x=1$$

Now,

$$LHS=\sum\frac{a}{a^5+1}\leq\sum\frac{1}{a^3+1}\overset{(*)}{\leq}\frac{3}{2}$$

$$(*)\leftrightarrow\frac{1}{a^3+1}+\frac{1}{b^3+1}+\frac{1}{c^3+1}\leq\frac{3}{2}$$

$$\leftrightarrow 2\sum(a^3+1)(b^3+1)\leq 3(a^3+1)(b^3+1)(c^3+1)$$

$$\leftrightarrow 3a^3b^3c^3+a^3b^3+b^3c^3+c^3a^3-a^3-b^3-c^3-3\geq 0$$

$$\overset{abc=1}{\leftrightarrow}\ 3+a^3b^3+b^3c^3+c^3a^3-a^3-b^3-c^3-3\geq 0$$

$$\leftrightarrow a^3b^3+b^3c^3+c^3a^3-a^3-b^3-c^3\geq 0$$

$$\overset{u=a^3,v=b^3,w=c^3}{\leftrightarrow}\quad uv+vw+wu\overset{(**)}{\geq}u+v+w$$

$$\overset{uvw=1}{\leftrightarrow}\ uv+vw+wu\geq\sqrt[3]{uvw}(u+v+w)$$

$$\leftrightarrow (uv+vw+wu)^3 \geq uvw(u+v+w)^3$$

$$\leftrightarrow -(uw-v^2)(u^2-vw)(uv-w^2) \geq 0$$

$$\leftrightarrow (v^2-uw)(u^2-vw)(uv-w^2) \geq 0$$

$$\overset{uvw=1}{\leftrightarrow} (v^3-1)(u^3-1)(1-w^3) \geq 0$$

Which is clearly true because:

$$uvw=1, u,v,w>0 \rightarrow (uvw)^3 = u^3v^3w^3 = 1$$

$$\overset{Dirichlet}{\rightarrow} (u^3-1)(v^3-1)(w^3-1) \leq 0.$$

$\rightarrow (**)$ is true . Proved.

6.145 If $x,y,z>0, xy+yz+zx=3, n\in\mathbb{N}, n\geq 2$ then:

$$\sqrt[n]{\frac{2x^{n+1}}{(y+z)^{3n+1}}}+\sqrt[n]{\frac{2y^{n+1}}{(z+x)^{3n+1}}}+\sqrt[n]{\frac{2z^{n+1}}{(x+y)^{3n+1}}} \geq \frac{3}{8}$$

Marin Chirciu

Solution(George Florin Şerban)

$$3 = xy+yz+zx \overset{Am-Gm}{\geq} 3\sqrt[3]{xy\cdot yz\cdot zx} \Rightarrow 3 \geq 3\sqrt[3]{(xyz)^2} \Rightarrow xyz \leq 1$$

$$\textit{Let}: f:(0,\infty)\rightarrow\mathbb{R}, f(x) = x^{-\frac{1}{n}}, f'(x) = -\frac{1}{n}x^{-\frac{1}{n}-1},$$

$$f''(x) = \frac{1}{n}\left(\frac{1}{n}+1\right)x^{-\frac{1}{n}-2} > 0 \Rightarrow f - \textit{is convexe.}$$

$$f\left(\frac{a+b+c}{3}\right) \leq \frac{f(a)+f(b)+f(c)}{3} \Rightarrow \sum_{cyc} f(a) \geq 3\cdot f\left(\frac{\sum a}{3}\right)$$

$$\sum_{cyc}\sqrt[n]{\frac{2x^{n+1}}{(y+z)^{3n+1}}} = \sum_{cyc}\left(\frac{2x^{n+1}}{(y+z)^{3n+1}}\right)^{\frac{1}{n}} = \sum_{cyc}\left(\frac{(y+z)^{3n+1}}{2x^{n+1}}\right)^{-\frac{1}{n}}$$

$$\geq 3\cdot\left[\frac{1}{3}\sum_{cyc}\left(\frac{(y+z)^{3n+1}}{2x^{n+1}}\right)\right]^{-\frac{1}{n}} \overset{Am-Gm}{\geq} 3\cdot\left(\sqrt[3]{\prod\frac{(y+z)^{3n+1}}{2x^{n+1}}}\right)^{-\frac{1}{n}}$$

$$\overset{Am-Gm}{\geq} 3\cdot\left(\sqrt[3]{\prod\frac{\left(2\sqrt{yz}\right)^{3n+1}}{2x^{n+1}}}\right)^{-\frac{1}{n}} = 3\cdot\left(\sqrt[3]{\frac{8^{3n+1}\cdot(xyz)^{3n+1}}{8\cdot(xyz)^{n+1}}}\right)^{-\frac{1}{n}}$$

$$= 3\cdot\left(\sqrt[3]{8^{3n}\cdot(xyz)^{2n}}\right)^{-\frac{1}{n}} = \frac{3}{\sqrt[3n]{8^{3n}\cdot(xyz)^{2}}} \geq \frac{3}{8\sqrt[3]{1^{2}}} = \frac{8}{3}$$

6.146 If $a,b,c>0$ then:

$$\sum_{cyc}\frac{c+\sqrt{ab}}{\sqrt{ab}(a+b+2c)} \geq \frac{1}{a+b}+\frac{1}{b+c}+\frac{1}{c+a}$$

Daniel Sitaru

Solution (Tran Hong)

$$\sum_{cyc}\frac{c+\sqrt{ab}}{\sqrt{ab}(a+b+2c)} =$$

$$= \frac{c+\sqrt{ab}}{\sqrt{ab}(a+b+2c)}+\frac{b+\sqrt{ac}}{\sqrt{ac}(a+c+2b)}+\frac{a+\sqrt{bc}}{\sqrt{bc}(b+c+2a)}$$

$$= \frac{1}{\sqrt{ab}(a+b+2c)}+\frac{1}{\sqrt{ac}(a+c+2b)}+\frac{1}{\sqrt{bc}(b+c+2a)}+$$

$$+\frac{c}{\sqrt{ab}(a+b+2c)}+\frac{b}{\sqrt{ac}(a+c+2b)}+\frac{a}{\sqrt{bc}(b+c+2a)} = \Omega$$

$$\Rightarrow \Omega \overset{Am-Gm}{\geq} \left(\frac{1}{a+b+2c}+\frac{1}{b+c+2a}+\frac{1}{a+c+2b}\right)$$

$$+\left(\frac{\frac{2c}{a+b}}{a+b+2c}+\frac{\frac{2b}{a+c}}{a+c+2b}+\frac{\frac{2a}{b+c}}{b+c+2a}\right)$$

$$=\frac{1+\frac{2c}{a+b}}{a+b+2c}+\frac{1+\frac{2b}{a+c}}{a+c+2b}+\frac{1+\frac{2a}{b+c}}{b+c+2a}$$

$$=\frac{\frac{a+b+2c}{a+b}}{a+b+2c}+\frac{\frac{a+c+2b}{a+c}}{a+c+2b}+\frac{\frac{b+c+2a}{b+c}}{b+c+2a}$$
$$=\frac{1}{a+b}+\frac{1}{b+c}+\frac{1}{c+a}$$

6.147 Let $x,y,z\in(0,\infty)$, prove:

$$\frac{[x]+1}{\sqrt{y^2+[z^2]}}+\frac{[y]+1}{\sqrt{z^2+[x^2]}}+\frac{[z]+1}{\sqrt{x^2+[y^2]}}>2$$

Jalil Hajimir

Solution (George Florin Şerban)

$$t-1<[t]\le t,\forall t\in\mathbb{R}$$
$$z^2-1<[z^2]\le z^2 \textit{ and analogs.}$$

$$\sum_{cyc}\frac{[x]+1}{\sqrt{y^2+[z^2]}}>\sum_{cyc}\frac{[x]+1}{\sqrt{y^2+z^2}}>\sum_{cyc}\frac{x}{\sqrt{y^2+z^2}}=\sum_{cyc}\sqrt{\frac{x^2}{y^2+z^2}}$$

$$=\sum_{cyc}\sqrt{\frac{x^2}{y^2+z^2}\cdot 1}\overset{Am-Hm}{\ge}\sum_{cyc}\frac{\frac{2x^2}{y^2+z^2}}{\frac{x^2}{y^2+z^2}+1}$$

$$=\sum_{cyc}\frac{2x^2}{x^2+y^2+z^2}=2$$

6.148 If $x,y,z>0,xyz=1,n\in\mathbb{N}-\{0\}$ then:

$$x^n\sqrt{\frac{2}{y+z}}+y^n\sqrt{\frac{2}{z+x}}+z^n\sqrt{\frac{2}{x+y}}\ge 3$$

Marin Chirciu

Solution(Marian Ursărescu)

$$\sqrt{\frac{2}{y+z}}=\sqrt{\frac{2}{y+z}\cdot 1}\geq\frac{2}{\frac{y+z}{2}+1}=\frac{4}{y+z+2}$$

We must show: $\frac{x^n}{y+z+2}+\frac{y^n}{z+x+2}+\frac{z^n}{x+y+2}\geq\frac{3}{4};\ \ (1)$

From Holder inequality we have:

$$\frac{x^n}{y+z+2}+\frac{y^n}{z+x+2}+\frac{z^n}{x+y+2}\geq\frac{(x+y+z)^n}{3^{n-2}\cdot 2(x+y+z+3)};\ \ (2)$$

From (1),(2) we must show:

$$\frac{(x+y+z)^n}{3^{n-2}\cdot 2(x+y+z+3)}\geq\frac{3}{4}\Leftrightarrow 2(x+y+z)^n\geq 3^{n-1}(x+y+z+3);\ \ (3)$$

Because $xyz=1\Rightarrow\exists a,b,c>0$ *such that* $x=\frac{a^2}{bc},y=\frac{b^2}{ca},z=\frac{c^2}{ab}\overset{(3)}{\Leftrightarrow}$

$$2\left(\frac{a^2}{bc}+\frac{b^2}{ca}+\frac{c^2}{ab}\right)^n\geq 3^{n-1}\left(\frac{a^2}{bc}+\frac{b^2}{ca}+\frac{c^2}{ab}+3\right)\Leftrightarrow$$

$$2\left(\frac{a^3+b^3+c^3}{abc}\right)^n\geq 3^{n-1}\left(\frac{a^3+b^3+c^3+3abc}{abc}\right)\Leftrightarrow$$

$$2(a^3+b^3+c^3)^n\geq 3^{n-1}(abc)^{n-1}(a^3+b^3+c^3)+3^n(abc)^n;\ \ (4)$$

But $(a^3+b^3+c^3)^n\geq(3abc)^n;(5)$ *and* $(a^3+b^3+c^3)^{n-1}\geq 3^{n-1}(abc)^{n-1}$

$$\Rightarrow(a^3+b^3+c^3)^n\geq 3^{n-1}(abc)^{n-1}(a^3+b^3+c^3);\ \ (6)$$

From (5),(6) we get (4) true.

6.149 If $a,b,c>0$ then:

$$\frac{1}{a+ab+b}+\frac{1}{b+bc+c}+\frac{1}{c+ca+a}\leq\sqrt{\frac{a^2+b^2+c^2}{3a^2b^2c^2}}$$

Daniel Sitaru

Solution(Florică Anastase)

$$a+ab+b\geq 3\sqrt[3]{a^2b^2}\to\frac{1}{a+ab+b}\leq\frac{1}{3\sqrt[3]{a^2b^2}}\to$$

$$\frac{1}{a+ab+b}+\frac{1}{b+bc+c}+\frac{1}{c+ca+a}\le\frac{1}{3}\left(\frac{1}{\sqrt[3]{a^2b^2}}+\frac{1}{\sqrt[3]{b^2c^2}}+\frac{1}{\sqrt[3]{c^2a^2}}\right)$$

$$\le\sqrt{\frac{1}{3}\left(\frac{1}{a^2b^2}+\frac{1}{b^2c^2}+\frac{1}{c^2a^2}\right)}=\sqrt{\frac{a^2+b^2+c^2}{3a^2b^2c^2}}$$

6.150 If $a,b,c>0, abc=4^n, n\in\mathbb{N}^*$ then:

$$\sum_{cyc}\frac{(n+1)(b^{2n+3}+c^{2n+3})+a^{2n+3}}{(b+c)^{2n}}\ge 3(2n+3)$$

Marin Chirciu

Solution (Tran Hong)

$$\text{For } n\in\mathbb{N}^*, a,b,c>0: b^{2n+3}+c^{2n+3}\ge\frac{(b+c)^{2n+3}}{2^{2n+2}}\Rightarrow$$

$$\frac{(n+1)(b^{2n+3}+c^{2n+3})}{(b+c)^{2n}}\ge\frac{n+1}{4^{n+1}}\cdot(b+c)^3$$

$$\sum_{cyc}\frac{(n+1)(b^{2n+3}+c^{2n+3})}{(b+c)^{2n}}\ge\frac{n+1}{4^{n+1}}\cdot[(a+b)^3+(b+c)^3+(c+a)^3]\ge$$

$$\ge\frac{n+1}{4^{n+1}}\cdot\frac{8}{9}\cdot(a+b+c)^3\ge\frac{n+1}{4^{n+1}}\cdot\frac{8}{9}\cdot 27abc=\frac{n+1}{4^{n+1}}\cdot\frac{8}{9}\cdot 4^n$$

$$=6(n+1);(*)$$

$$\text{Suppose}: a\ge b\ge c\Rightarrow a^3\ge b^3\ge c^3;\ \frac{a}{b+c}\ge\frac{b}{c+a}\ge\frac{c}{a+b}\Rightarrow$$

$$\left(\frac{a}{b+c}\right)^{2n}\ge\left(\frac{b}{c+a}\right)^{2n}\ge\left(\frac{c}{a+b}\right)^{2n}$$

$$\Rightarrow\frac{a^{2n+3}}{(b+c)^{2n}}+\frac{b^{2n+3}}{(c+a)^{2n}}+\frac{c^{2n+3}}{(a+b)^{2n}}$$

$$\overset{Cebyshev}{\ge}\frac{1}{3}(a^3+b^3+c^3)\left(\left(\frac{a}{b+c}\right)^{2n}+\left(\frac{b}{c+a}\right)^{2n}+\left(\frac{c}{a+b}\right)^{2n}\right)$$

$$\ge\frac{1}{3}\cdot(3abc)\cdot\left(\frac{\left(\frac{a}{b+c}+\frac{b}{c+a}+\frac{c}{a+b}\right)^{2n}}{3^{2n-1}}\right)\overset{Nesbitt}{\ge}4^n\cdot\frac{\left(\frac{3}{2}\right)^{2n}}{3^{2n-1}}=3;\quad(**)$$

$$(*)+(**)\Rightarrow\sum_{cyc}\frac{(n+1)(b^{2n+3}+c^{2n+3})+a^{2n+3}}{(b+c)^{2n}}=6(n+1)+3$$

$$\geq 3(2n+3)$$

6.151 If $x,y,z>0, xy+yz+zx=3$ then:

$$\sqrt{\frac{2x}{(y+z)^5}}+\sqrt{\frac{2y}{(z+x)^5}}+\sqrt{\frac{2z}{(x+y)^5}}\geq\frac{3}{4}$$

Marin Chirciu

Solution (Tran Hong)

With $x,y,z>0$, let: $t=xy+yz+zx\geq\sqrt{3(xy+yz+zx)}=3$

$$\sqrt{\frac{2x}{(y+z)^5}}=\frac{1}{(y+z)^2}\cdot\sqrt{\frac{2x}{y+z}}=\frac{1}{(y+z)^2}\cdot\sqrt{1\cdot\frac{2x}{y+z}}\overset{Am-Hm}{\geq}$$

$$\geq\frac{1}{(y+z)^2}\cdot\frac{2}{1+\frac{1}{\frac{2x}{y+z}}}=\frac{1}{(y+z)^2}\cdot\frac{2}{1+\frac{y+z}{2x}}=$$

$$=\frac{4x}{(y+z)^2(2x+y+z)}=\frac{4\left(\frac{x}{y+z}\right)^2}{2x^2+xy+xz}$$

Similarly:

$$\sqrt{\frac{2y}{(z+x)^5}}\geq\frac{4\left(\frac{y}{z+x}\right)^2}{2y^2+yz+yz}\text{ and }\sqrt{\frac{2z}{(x+y)^5}}\geq\frac{4\left(\frac{z}{x+y}\right)^2}{2z^2+zx+zy}$$

$$LHS=\sum_{cyc}\sqrt{\frac{2x}{(y+z)^5}}\geq$$

$$\geq 4\left[\frac{\left(\frac{x}{y+z}\right)^2}{2x^2+xy+xz}+\frac{\left(\frac{y}{z+x}\right)^2}{2y^2+yz+yz}+\frac{\left(\frac{z}{x+y}\right)^2}{2z^2+zx+zy}\right]$$

$$\overset{Bergstrom}{\geq} 4\cdot\frac{\left(\frac{x}{y+z}+\frac{y}{z+x}+\frac{z}{x+y}\right)^2}{2(x^2+y^2+z^2+xy+yz+zx)}$$

$$=\frac{2\left(\frac{x}{y+z}+\frac{y}{z+x}+\frac{z}{x+y}\right)^2}{(x^2+y^2+z^2+xy+yz+zx)}=\frac{2\left(\frac{x}{y+z}+\frac{y}{z+x}+\frac{z}{x+y}\right)^2}{(x+y+z)^2-3}$$

$$\frac{x}{y+z}+\frac{y}{z+x}+\frac{z}{x+y}=\frac{x^2}{xy+xz}+\frac{y^2}{yz+yx}+\frac{z^2}{zx+zy}\overset{Bergstrom}{\geq}$$

$$\geq\frac{(x+y+z)^2}{2(xy+yz+zx)}=\frac{(x+y+z)^2}{6}$$

So, we need to prove: $\frac{2\cdot\frac{t^4}{6^2}}{t^2-3}\geq\frac{3}{4}\Leftrightarrow 2t^4\geq 27(t^2-3)\Leftrightarrow 2t^4-27t^2+81\geq 0$

$\Leftrightarrow (t^2-9)(2t^2-9)\geq 0$ true for $t\geq 3\Leftrightarrow t^2-9\geq 0; 2t^2-9\geq 9>0$

Proved.

Equality$\Leftrightarrow x=y=z=1.$

6.152 If $a,b,c>0,\mu\leq 1$ then:

$$\frac{a}{b+c}+\frac{b}{c+a}+\frac{c}{a+b}+\mu\cdot\sqrt{\frac{2abc}{(a+b)(b+c)(c+a)}}\geq\frac{\mu+3}{2}$$

Marin Chirciu

Solution(Tran Hong)

$$\frac{a}{b+c}+\frac{b}{c+a}+\frac{c}{a+b}+\mu\cdot\sqrt{\frac{2abc}{(a+b)(b+c)(c+a)}}\geq\frac{\mu+3}{2}$$

$$\frac{a}{b+c}+\frac{b}{c+a}+\frac{c}{a+b}\overset{(*)}{\geq}\frac{\mu+3}{2}-\mu\cdot\sqrt{\frac{2abc}{(a+b)(b+c)(c+a)}}$$

More, by Schur's inequality:

$$\frac{a}{b+c}+\frac{b}{c+a}+\frac{c}{a+b}+\frac{4abc}{(a+b)(b+c)(c+a)}\geq 2$$

$$\frac{a}{b+c}+\frac{b}{c+a}+\frac{c}{a+b}\overset{(**)}{\geq}2-\frac{4abc}{(a+b)(b+c)(c+a)}$$

From $(*),(**)$ *we need to prove:*

$$2-\frac{4abc}{(a+b)(b+c)(c+a)}\geq\frac{\mu+3}{2}-\mu\cdot\sqrt{\frac{2abc}{(a+b)(b+c)(c+a)}}$$

$$\frac{1-\mu}{2}\overset{(1)}{\geq}2\cdot\frac{2abc}{(a+b)(b+c)(c+a)}-\mu\cdot\sqrt{\frac{2abc}{(a+b)(b+c)(c+a)}}$$

For $a,b,c>0\Rightarrow(a+b)(b+c)(c+a)\overset{Am-Gm}{\geq}8abc$ *(Cesaro)*

$$\Rightarrow 0<\frac{2abc}{(a+b)(b+c)(c+a)}\leq\frac{1}{4}$$

Let: $t=\frac{2abc}{(a+b)(b+c)(c+a)}\Rightarrow 0<t\leq\frac{1}{2}$

$$(1)\Leftrightarrow 2t^2-\mu t-\left(\frac{1-\mu}{2}\right)\leq 0\Leftrightarrow\frac{1}{2}(1-2t)\big(\mu-(1+2t)\big)\leq 0$$

Which is clearly true because:

$0<t\leq\frac{1}{2};\ \mu\leq 1\Rightarrow 1+2t>1\geq\mu\Rightarrow\mu-(1+2t)<0$.*Proved.*

Equality holds if $a=b=c$.

6.153 If $a,b,c,d>1, abcd=e^4$ then:

$$\frac{\log\left(\frac{e^2}{a}\right)\cdot\log\left(\frac{e^2}{b}\right)\cdot\log\left(\frac{e^2}{c}\right)\cdot\log\left(\frac{e^2}{d}\right)}{\log(ab)\cdot\log(bc)\cdot\log(cd)\cdot\log(da)}\leq\frac{1}{16}$$

Daniel Sitaru

Solution (Rahim Shahbazov)

$$abcd=e^4\rightarrow\log a+\log b+\log c+\log d=4$$

$$a,b,c,d>1\rightarrow\log a,\log b,\log c,\log d>0$$

Let: $x=\log a, y=\log b, z=\log c, t=\log d\rightarrow$

$$\frac{(2-x)(2-y)(2-z)(2-t)}{(x+y)(y+z)(z+t)(t+x)}\leq\frac{1}{16}$$

$$\rightarrow (x+y)(y+z)(z+t)(t+x) \ge (4-2x)(4-2y)(4-2z)(4-2t) \rightarrow$$

$$(x+y)(y+z)(z+t)(t+x) \ge$$

$$\ge (z+y+t-x)(z+x+t-y)(z+x+y-t)(x+y+t-z)$$

$$\begin{cases} y+z+t-x = A > 0 \\ z+x+t-y = B > 0 \\ z+x+y-t = C > 0 \\ x+y+t-z = D > 0 \end{cases} \rightarrow \begin{cases} 2(z+t) = A+B \\ 2(x+y) = C+D \\ 2(x+t) = B+D \\ 2(y+z) = A+C \end{cases}$$

$\rightarrow (A+B)(C+D)(B+D)(A+C) \ge 16ABCD$ *true from Am-Gm.*

6.154 If $x, y, z > 0, x+y+z = 3$ then:

$$3x^4 + 5y^4 + 6z^4 \ge \frac{81}{\left(\frac{1}{\sqrt[3]{3}} + \frac{1}{\sqrt[3]{5}} + \frac{1}{\sqrt[3]{6}}\right)^3}$$

Rahim Shahbazov

Solution(Daniel Sitaru)

$$3x^4 + 5y^4 + 6z^4 = \frac{x^4}{\frac{1}{3}} + \frac{y^4}{\frac{1}{5}} + \frac{z^4}{\frac{1}{6}} = \frac{x^4}{\left(\frac{1}{\sqrt[3]{3}}\right)^3} + \frac{y^4}{\left(\frac{1}{\sqrt[3]{5}}\right)^3} + \frac{z^4}{\left(\frac{1}{\sqrt[3]{6}}\right)^3} \ge$$

$$\overset{RADON}{\ge} \frac{(x+y+z)^4}{\left(\frac{1}{\sqrt[3]{3}} + \frac{1}{\sqrt[3]{5}} + \frac{1}{\sqrt[3]{6}}\right)^3} = \frac{81}{\left(\frac{1}{\sqrt[3]{3}} + \frac{1}{\sqrt[3]{5}} + \frac{1}{\sqrt[3]{6}}\right)^3}$$

Equality holds for:

$$\begin{cases} x = \dfrac{3}{\sqrt[3]{3}\left(\frac{1}{\sqrt[3]{3}} + \frac{1}{\sqrt[3]{5}} + \frac{1}{\sqrt[3]{6}}\right)} \\ y = \dfrac{3}{\sqrt[3]{5}\left(\frac{1}{\sqrt[3]{3}} + \frac{1}{\sqrt[3]{5}} + \frac{1}{\sqrt[3]{6}}\right)} \\ z = \dfrac{3}{\sqrt[3]{6}\left(\frac{1}{\sqrt[3]{3}} + \frac{1}{\sqrt[3]{5}} + \frac{1}{\sqrt[3]{6}}\right)} \end{cases}$$

6.155 If $a,b,c,d \in \mathbb{R}$ then:

$$4(ad-bc)^6+4(ac+bd)^6 \geq (a^2+b^2)^3(c^2+d^2)^3$$

Daniel Sitaru

Solution(George Florin Şerban)

Applying Lagrange identity: $(ad-bc)^2+(ac+bd)^2=(a^2+b^2)(c^2+d^2)$ we have:

$$(ad-bc)^6+(ac+bd)^6=\frac{[(ad-bc)^2]^3}{1}+\frac{[(ac+bd)^2]^3}{1}\overset{Holder}{\geq}$$

$$\geq \frac{[(ad-bc)^2+(ac+bd)^2]^3}{(1+1)\cdot 2^{3-2}}=\frac{(a^2+b^2)^3(c^2+d^2)^3}{4}$$

$$4(ad-bc)^6+4(ac+bd)^6 \geq (a^2+b^2)^3(c^2+d^2)^3$$

6.156 If $a,b,c,d \in \mathbb{R}$ then:

$$(ad-bc)^8(a^2+b^2)(c^2+d^2)+(ac+bd)^{10}$$
$$\leq (a^2+b^2)^5(c^2+d^2)^5$$

Daniel Sitaru

Solution(Ravi Prakash)

Case 1: One of a,b,c,d is 0

Let $a=0$.The inequality becomes:

$$b^8c^8b^2(c^2+d^2) \leq b^{10}(c^2+d^2)^5$$

$$LHS=b^{10}(c^{10}+c^8d^2) \leq b^{10}(c^{10}+5c^8d^2+\dots+d^{10})=$$
$$=b^{10}(c^2+d^2)=RHS$$

Similarly, for the other cases like $b=0$ or $c=0$ or $d=0$.

Cases 2: $abcd \neq 0$.

Let $a=r_1cos\theta, b=r_1sin\theta, c=r_2cos\varphi, d=r_2sin\varphi$ where $r_1,r_2>0$ and $0<\theta,\varphi<2\pi$

Now, $LHS=r_1^{10}r_2^{10}sin^8(\theta-\varphi)+r_1^{10}r_2^{10}cos^{10}(\theta-\varphi)=$

$$=r_1^{10}r_2^{10}[sin^8(\theta-\varphi)+cos^{10}(\theta-\varphi)]=$$

$$= r_1^{10} r_2^{10} [sin^2(\theta - \varphi) + cos^2(\theta - \varphi)] =$$

$$= r_1^{10} r_2^{10} = (a^2 + b^2)^5 (c^2 + d^2)^5 = RHS$$

6.157 If $\mathbf{1 < a < b \leq e}$ then:

$$\mathbf{125^a \cdot (4a + b)^{a+4b} \leq 125^b \cdot (a + 4b)^{4a+b}}$$

Daniel Sitaru

Solution(Florică Anastase)

For $a < b \Rightarrow 125^a < 125^b$; (1)

Let be the function: $f: (1, \infty) \to \mathbb{R}, f(x) = \frac{logx}{x}, f'(x) = \frac{1-logx}{x^2}$

$f'(x) = 0 \Leftrightarrow 1 - logx = 0 \Leftrightarrow x = e$ and how $logx < 1, \forall x \in (1, e] \Rightarrow 1 - logx > 0 \Rightarrow f(x)$ −increasing on $(1, e]$.

$$a < b \leq e \Rightarrow 3a < 3b \Rightarrow 5 < 3a + (a + b) < (a + b) + 3b < 5e$$

$\Rightarrow 1 < \frac{4a+b}{5} < \frac{a+4b}{5} < e$ then

$$f\left(\frac{4a + b}{5}\right) < f\left(\frac{a + 4b}{5}\right) \Rightarrow$$

$$\frac{log(4a + b) - log5}{4a + b} < \frac{log(a + 4b) - log5}{a + 4b} \Leftrightarrow$$

$$(a + 4b)log(4a + b) + 3alog5 < (4a + b)log(a + 4b) + 3blog5 \Leftrightarrow$$

$$log[5^{3a}(4a + b)^{a+4b}] < log[5^{3b}(a + 4b)^{4a+b}] \Leftrightarrow$$

$$5^{3a}(4a + b)^{a+4b} < 5^{3b}(a + 4b)^{4a+b}; (2)$$

From (1),(2) we get:

$$125^a \cdot (4a + b)^{a+4b} \leq 125^b \cdot (a + 4b)^{4a+b}$$

6.158 If $x, y, z \in \mathbb{R}$ then:

$$\frac{(x^{12}+x^6+1)(y^{24}+y^{12}+1)(z^{36}+z^{18}+1)}{(x^8+1)(y^{16}+1)(z^{24}+1)} > x^2y^4z^6$$

Daniel Sitaru

Solution(Tran Hong)

For $a \in \mathbb{R}$ *we have:* $\frac{a^6+a^3+1}{a^4+1} > a \Leftrightarrow a^6+a^3+1 > a(a^4+1) \Leftrightarrow$

$$a^6-a^5+a^3-a+1>0 \Leftrightarrow (a^2-a+1)(a^4-a^2+1)>0 \Leftrightarrow$$

$$\left[\left(a-\frac{1}{2}\right)^2+\frac{3}{4}\right]\left[\left(a^2-\frac{1}{2}\right)^2+\frac{3}{4}\right]>0, \forall a \in \mathbb{R} \text{ true.}$$

Now, choose: $a = x^2; b = y^4; c = z^6$ *we get:*

$$\frac{(x^{12}+x^6+1)(y^{24}+y^{12}+1)(z^{36}+z^{18}+1)}{(x^8+1)(y^{16}+1)(z^{24}+1)} > x^2y^4z^6$$

6.159 If $0 < a \le b < 1$ then:

$$\sin\left(\frac{3a+b+2}{4}\right)\sin\left(\frac{a+3b+6}{4}\right) \le \sin\left(\frac{a+3b+2}{4}\right)\sin\left(\frac{3a+b+6}{4}\right)$$

Daniel Sitaru

Solution(Adrian Popa)

$$\sin\alpha\sin\beta = \frac{\cos(\alpha-\beta)-\cos(\alpha+\beta)}{2} \Rightarrow$$

$$Lhs = \sin\left(\frac{3a+b+2}{4}\right)\sin\left(\frac{a+3b+6}{4}\right) =$$

$$= \frac{\cos\left(\frac{3a+b+2}{4}-\frac{a+3b+6}{4}\right)-\cos\left(\frac{3a+b+2}{4}+\frac{a+3b+6}{4}\right)}{2} =$$

$$= \frac{\cos\left(\frac{2a-2b-4}{4}\right)-\cos\left(\frac{4a+4b+8}{4}\right)}{2} =$$

$$=\frac{cos\left(\frac{a-b-2}{2}\right)-cos(a+b+2)}{2}$$

$$Rhs=sin\left(\frac{a+3b+2}{4}\right)sin\left(\frac{3a+b+6}{4}\right)=$$

$$=\frac{cos\left(\frac{a+3b+2}{4}-\frac{3a+b+6}{4}\right)-cos\left(\frac{a+3b+2}{4}+\frac{3a+b+6}{4}\right)}{4}=$$

$$=\frac{cos\left(\frac{-2a+2b-4}{4}\right)-cos\left(\frac{4a+4b+8}{4}\right)}{2}=$$

$$=\frac{cos\left(\frac{-a+b-2}{2}\right)-cos(a+b+2)}{2}$$

We must show that:

$$\frac{cos\left(\frac{a-b-2}{2}\right)-cos(a+b+2)}{2}\le\frac{cos\left(\frac{-a+b-2}{2}\right)-cos(a+b+2)}{2}\Leftrightarrow$$

$$cos\left(\frac{a-b-2}{2}\right)\le cos\left(\frac{-a+b-2}{2}\right)\Leftrightarrow cos\left(\frac{a-b}{2}-1\right)$$

$$\le cos\left(-\frac{a-b}{2}-1\right)\Leftrightarrow$$

$$cos\left(-\frac{b-a}{2}-1\right)\le cos\left(-\frac{b-a}{2}+1\right)\Leftrightarrow cos\left(\frac{b-a}{2}+1\right)$$

$$\le cos\left(1-\frac{b-a}{2}\right)\overset{cosx\downarrow\left(0;\frac{\pi}{2}\right)}{\Longleftrightarrow}$$

$\frac{b-a}{2}+1\ge 1-\frac{b-a}{2}\Leftrightarrow b\ge a$ true.Proved.

6.160 If $a,b\ge 0$ then:

$$\sqrt{ab}+\sqrt[7]{\left(\frac{2ab}{a+b}\right)^7-\left(\sqrt{ab}\right)^7+\left(\frac{a+b}{2}\right)^7}\ge\frac{2ab}{a+b}+\frac{a+b}{2}$$

Daniel Sitaru

Solution(George Florin Şerban)

If $b=0\Rightarrow\frac{a}{2}\ge\frac{a}{2}$ true.

If $b\ne 0,\frac{a}{b}=t>0\Rightarrow$

$$\sqrt[7]{\left(\frac{2t}{t+1}\right)^7-\left(\sqrt{t}\right)^7+\left(\frac{t+1}{2}\right)^7}\ge\frac{2t}{t+1}-\sqrt{t}+\frac{t+1}{2}\Leftrightarrow$$

$$\left(\frac{2t}{t+1}\right)^7-\left(\sqrt{t}\right)^7+\left(\frac{t+1}{2}\right)^7\geq\left(\frac{2t}{t+1}-\sqrt{t}+\frac{t+1}{2}\right)^7\Leftrightarrow$$
$$\left(\frac{t+1}{2}\right)^7-\left(\sqrt{t}\right)^7\geq\left(\frac{2t}{t+1}-\sqrt{t}+\frac{t+1}{2}\right)^7-\left(\frac{2t}{t+1}\right)^7\Leftrightarrow$$
$$\left(\frac{t+1}{2}-\sqrt{t}\right)\left[\left(\frac{t+1}{2}\right)^6+\left(\frac{t+1}{2}\right)^5\sqrt{t}+\cdots+\left(\sqrt{t}\right)^6\right]\geq$$
$$\geq\left(\frac{2t}{t+1}-\sqrt{t}+\frac{t+1}{2}-\frac{2t}{t+1}\right)\left[\left(\frac{2t}{t+1}-\sqrt{t}+\frac{t+1}{2}\right)^6\right.$$
$$\left.+\left(\frac{2t}{t+1}-\sqrt{t}+\frac{t+1}{2}\right)^5\frac{2t}{t+1}+\cdots+\left(\frac{2t}{t+1}\right)^6\right]$$
$$\frac{t+1}{2}\geq\sqrt{t}\Rightarrow\frac{t+1}{2}-\sqrt{t}\geq 0$$
$$\frac{t+1}{2}-\sqrt{t}\geq\frac{2t}{t+1}-\sqrt{t}+\frac{t+1}{2}-\frac{2t}{t+1}=\frac{t+1}{2}-\sqrt{t}\geq 0$$
$$\frac{2t}{t+1}\leq\sqrt{t}\leq\frac{t+1}{2}$$
$$\left(\frac{t+1}{2}\right)^6\geq\left(\frac{2t}{t+1}-\sqrt{t}+\frac{t+1}{2}\right)^6\Rightarrow\frac{t+1}{2}\geq\frac{2t}{t+1}-\sqrt{t}+\frac{t+1}{2}$$
$\Rightarrow\sqrt{t}\geq\frac{2t}{t+1}$ true by Gm-Hm.
$$\left(\frac{t+1}{2}\right)^5\geq\left(\frac{2t}{t+1}-\sqrt{t}+\frac{t+1}{2}\right)^5\Rightarrow\frac{t+1}{2}\geq\frac{2t}{t+1}-\sqrt{t}+\frac{t+1}{2}$$
$\Rightarrow\sqrt{t}\geq\frac{2t}{t+1}$ true by Gm-Hm.

$\left(\sqrt{t}\right)^6\geq\left(\frac{2t}{t+1}\right)^6\Rightarrow\sqrt{t}\geq\frac{2t}{t+1}$ true by Gm-Hm.

6.161 If $a,b,c>0,abc=1$ then:

$$\sum_{cyc}\frac{(a^{10}+b^{10})(a^9+b^9)}{(a^4+b^4)(a^3+b^3)}\geq 3$$

Daniel Sitaru

Solution(Sanong Huayrerai)

$$\sum_{cyc}\frac{(a^{10}+b^{10})(a^9+b^9)}{(a^4+b^4)(a^3+b^3)}\geq$$

$$\geq \sum_{cyc} \frac{(a^4+b^4)(a^3+b^3)(a^6+b^6)(a^6+b^6)}{(a^4+b^4)(a^3+b^3)} \geq \sum_{cyc} \frac{(a^6+b^6)^2}{4} \geq$$

$$\geq \frac{\left(\sum \frac{a^6+b^6}{2}\right)^2}{3} = \frac{(a^6+b^6+c^6)^2}{3} \geq 3$$

If $abc = 1 \Rightarrow a+b+c \geq 3 \Rightarrow a^6+b^6+c^6 \geq 3$

6.162 If $a, b, c > 0$; $abc = 1$ then:

$$\frac{7-6a}{2+a^2} + \frac{7-6b}{2+b^2} + \frac{7-6c}{2+c^2} \geq 1$$

Jalil Hajimir

Solution(Michael Sterghiou)

$$\frac{7-6a}{2+a^2} + \frac{7-6b}{2+b^2} + \frac{7-6c}{2+c^2} \geq 1; (1)$$

Let $(p, q, r) = (\sum a, \sum ab, \prod a), r = 1$. By expanding (1) we get:

$\frac{\sum(7-6a)(2+b^2)(2+c^2)}{\prod(2+a^2)} \geq 1$ which after same computation reduces to:

$$24p^2 - 12pq - 34p + 5q^2 - 54q + 111 \geq 0$$

Or $6(p-q)^2 + f(q) \geq 0$ where $f(q) = 18p^2 - 34p - q^2 - 54q + 111$

Note that:

$\sum ab^2 + \sum a^2b = pq - 3; \sum a^2b^2 = q^2 - 2p; \sum a^2 = p^2 - 2q$ as $r = 1$

$f(q)$ is a decreasing function of q. Assuming $a \leq b \leq c$ (WLOG)

Wish p fixed $f(q)$ becomes minimal when $a = b(\leq 1)$ in which case if is enough that $f(q) \geq 0$.

Wish $a = b = x, c = \frac{1}{x^2}, p = 2x + \frac{1}{x^2}, q = \frac{2}{x} + x^2$ and

$$f(q) \to f(x) = -\frac{1}{x^4}(x-1)^2(x^6 + 2x^5 - 15x^4 + 40x^3 - 16x^2 - 36x - 18) = -\frac{(x-1)^2}{x^4} \cdot \sigma(x), \text{ where}$$

$$\sigma(x) = x^4\left(\underbrace{x^2 + 2x - 3}_{<0}\right) + x\left(\underbrace{40x^2 - 16x - 24}_{<0}\right) - 12x^4 - 12x - 18 < 0$$

For $0 \le x \le 1$.Hence $g(x) \ge 0$.

With equality for $x = a = b = 1, c = 1$.Done.

6.163 If $a, b, c > 0, \sqrt{ab} + \sqrt{bc} + \sqrt{ca} = 12$ then:

$$\frac{\left(a + b + \sqrt{ab}\right)^3}{(a+b)^2} + \frac{\left(b + c + \sqrt{bc}\right)^3}{(b+c)^2} + \frac{\left(c + a + \sqrt{ca}\right)^3}{(c+a)^2} \ge 81$$

Daniel Sitaru

Solution(Sanong Huayrerai)

For $a, b, c > 0, \sqrt{ab} + \sqrt{bc} + \sqrt{ca} = 12$ we give: $a = x^2, b = y^2, c = z^2$

Hence we have: $\sqrt{ab} + \sqrt{bc} + \sqrt{ca} = xy + yz + zx = 12$ and

$$\frac{\left(a + b + \sqrt{ab}\right)^3}{(a+b)^2} + \frac{\left(b + c + \sqrt{bc}\right)^3}{(b+c)^2} + \frac{\left(c + a + \sqrt{ca}\right)^3}{(c+a)^2} =$$
$$= \frac{(x^2 + y^2 + xy)^3}{(x^2+y^2)^2} + \frac{(y^2 + z^2 + yz)^3}{(y^2+z^2)^2} + \frac{(z^2 + x^2 + zx)^3}{(z^2+x^2)^2} \ge 81$$
$$= \frac{27}{4}(\underbrace{xy + yz + zx}_{12})$$

Let's consider: $\frac{(x^2+y^2+xy)^3}{(x^2+y^2)^2} \ge \frac{27}{4}xy \Leftrightarrow (x^2 + y^2 + xy)^3 \ge \frac{27}{4}xy(x^2 + y^2)^2$

$$4(x^6 + y^6 + (xy)^3) + 12(x^4y^2 + x^5y + x^2y^4 + xy^5 + x^4y^2 + x^2y^4) + 24x^3y^3 \ge 27(x^5y + xy^5) + 24(xy)^3$$

$$4(x^6 + y^6) + 24(x^4y^2 + x^2y^4) \ge 26(xy)^3 + 15(x^5y + xy^5)$$

$$4[x^5(x-y) - y^5(x-y)] + 13[x^3y^2(x-y) - x^2y^3(x-y)] \ge 11[x^4y(x-y) - xy^4(x-y)]$$

$$4(x-y)^2(x^4+x^3y+x^2y^2+xy^3+y^4)+13(x-y)^2(xy)^2$$
$$\geq 11(x-y)^2(x^2+xy+y^2)$$

$$4(x^4+x^3y+x^2y^2+xy^3+y^4)\geq 11(x^2+xy+y^2); x\neq y$$
$$4(x^4+y^4)+6(xy)^2\geq 7(x^3y+xy^3) \text{ true.Then}$$

$$\frac{(y^2+z^2+yz)^3}{(y^2+z^2)^2}\geq \frac{27}{4}yz; \frac{(z^2+x^2+zx)^3}{(z^2+x^2)^2}\geq \frac{27}{4}zx$$

$$\frac{(x^2+y^2+xy)^3}{(x^2+y^2)^2}+\frac{(y^2+z^2+yz)^3}{(y^2+z^2)^2}+\frac{(z^2+x^2+zx)^3}{(z^2+x^2)^2}\geq 81$$
$$=\frac{27}{4}(\underbrace{xy+yz+zx}_{12})$$
$$\frac{\left(a+b+\sqrt{ab}\right)^3}{(a+b)^2}+\frac{\left(b+c+\sqrt{bc}\right)^3}{(b+c)^2}+\frac{\left(c+a+\sqrt{ca}\right)^3}{(c+a)^2}\geq 81$$

6.164 If $a,b\geq 0$ then:

$$\frac{(a+b)^3}{8}+\frac{8a^3b^3}{(a+b)^3}\geq ab\sqrt{ab}+\left(\frac{\left(\sqrt{a}-\sqrt{b}\right)^2}{2}+\frac{2ab}{a+b}\right)^3$$

Daniel Sitaru

Solution(George Florin Şerban)

$$x=\frac{a+b}{2}=M_a; y=\frac{2ab}{a+b}=M_h; z=\sqrt{ab}=M_g\Rightarrow$$

$$x^3+y^3\geq z^3+\left(\frac{a+b}{2}-\sqrt{ab}+\frac{2ab}{a+b}\right)^3$$

$$x^3+y^3\geq z^3+(x-z+y)^3$$

$$(x+y)(x^2-xy+y^2)\geq$$
$$\geq (z+x-z+y)[z^2-z(x-z+y)+(x-z+y)^2]$$

$$(x+y)(x^2-xy+y^2)-(z+x-z+y)[z^2-z(x-z+y)+(x-z+y)^2]\geq 0\Leftrightarrow$$

$$(x+y)[x^2-xy+y^2-z^2+z(x-z+y)+(x-z+y)^2]\geq 0; x,y>0$$
$$\Rightarrow x+y>0$$

$$x^2-xy+y^2-z^2+z(x-z+y)+(x-z+y)^2=(z-x)(y-z)\geq 0$$
true by $M_h=y\leq M_g=z\leq M_a=x$.

6.165 If $a,b,c>0, a+b+c=6, 0\leq\mu\leq 4$ then:

$$\frac{a^2}{\mu+a^2}+\frac{b^2}{\mu+b^2}+\frac{c^2}{\mu+c^2}\leq\frac{12}{\mu+4}$$

Marin Chirciu

Solution(Tran Hong)

For $0\leq\mu\leq 4$; $\sum_{cyc}\frac{a^2}{\mu+a^2}\leq\frac{12}{\mu+4}\Leftrightarrow\sum_{cyc}\frac{1}{\mu+a^2}\geq\frac{3}{4+\mu}; (1)$

We show that (1) is true.

$6^2=(a+b+c)^2\geq 3(ab+bc+ca)\Rightarrow ab+bc+ca<12\Rightarrow\exists\alpha\geq c\geq 0$ such that

$$ab+b\alpha+\alpha a=12.$$

From $\alpha\geq c\Rightarrow\frac{1}{\mu+c^2}\geq\frac{1}{\mu+\alpha^2}$.

So, we need to prove:

$$\frac{1}{\mu+a^2}+\frac{1}{\mu+b^2}+\frac{1}{\mu+\alpha^2}\geq\frac{3}{4+\mu}; (2)$$

In fact, without loss of generality, assume that: $a=min\{a,b,\alpha\}\Rightarrow$
$(a-b)(a-\alpha)\geq 0\Rightarrow a^2+b\alpha\geq ab+a\alpha+b\alpha=12$
Other, $a^2=a\cdot a\leq b\cdot\alpha\Rightarrow 12\leq 3b\alpha\Rightarrow 4\leq b\alpha$.

From the CBS inequality:

$$(ab+b\alpha+a\alpha)\left(\frac{1}{ab}+\frac{1}{b\alpha}+\frac{1}{a\alpha}\right)\geq 9\Leftrightarrow$$

$$12\left(\frac{1}{ab}+\frac{1}{b\alpha}+\frac{1}{a\alpha}\right)\geq 9\Leftrightarrow a+b+\alpha\geq\frac{3}{4}ab\alpha; (*)\text{.Now,}$$

$$\frac{1}{\mu+b^2}+\frac{1}{\mu+\alpha^2}-\frac{2}{b\alpha+\mu}=\frac{b\alpha-b^2}{(b^2+\mu)(b\alpha+\mu)}+\frac{b\alpha-\alpha^2}{(a^2+\mu)(b\alpha+\mu)}=$$

$$=\frac{b(\alpha-b)}{(b^2+\mu)(b\alpha+\mu)}-\frac{\alpha(\alpha-b)}{(a^2+\mu)(b\alpha+\mu)}=\frac{(\alpha-b)(b\alpha^2+b\mu-\alpha b^2-\alpha\mu)}{(a^2+\mu)(b^2+\mu)(b\alpha+\mu)}=$$

$$=\frac{(\alpha-b)^2(\alpha b-\mu)}{(a^2+\mu)(b^2+\mu)(b\alpha+\mu)}\overset{(3)}{\geq}0;\ (\text{because}(\alpha-b)^2\geq 0,\alpha b\geq 4\geq\mu)$$

$$\frac{1}{a^2+\mu}+\frac{2}{b\alpha+\mu}-\frac{3}{4+\mu}=$$

$$=\frac{(4+\mu)(b\alpha+\mu)+2(a^2+\mu)(4+\mu)-3(a^2+\mu)(b\alpha+\mu)}{(4+\mu)(a^2+\mu)(b\alpha+\mu)}$$

$$=\frac{(4b\alpha+4\mu+b\alpha\mu+\mu^2)+2(4a^2+a^2\mu+4\mu+\mu^2)-3(b\alpha a^2+\mu a^2+b\alpha\mu+\mu^2)}{(4+\mu)(a^2+\mu)(b\alpha+\mu)}$$

$$=\frac{4b\alpha+12\mu+8a^2-2b\alpha\mu-3b\alpha a^2-\mu a^2}{(4+\mu)(a^2+\mu)(b\alpha+\mu)}=$$

$$=\frac{\mu(12-2b\alpha-a^2)+(4b\alpha+8a^2-3b\alpha a^2)}{(4+\mu)(a^2+\mu)(b\alpha+\mu)}\overset{\substack{12-2b\alpha-a^2\leq 0\\ 0\leq\mu\leq 4}}{=}$$

$$=\frac{4(12-2b\alpha-a^2)+(4b\alpha+8a^2-b\alpha a^2)}{(4+\mu)(a^2+\mu)(b\alpha+\mu)}\overset{ab+bc+ca=12}{=}$$

$$=\frac{4a\left(a+b+\alpha-\frac{3}{4}ab\alpha\right)}{(4+\mu)(a^2+\mu)(b\alpha+\mu)}\overset{(4)}{\geq}0;\left(\text{from}(*)\right)$$

From (3),(4) result (2) is true then (1) is true.Proved.

6.166 If $a,b,c>0,a+b+c=3,\mu\geq\frac{3}{2}$ then:

$$\sum_{cyc}\frac{a^2}{2(2\mu-1)b+\sqrt[3]{4(1+b^6)}}\geq\frac{3}{4\mu}$$

Marin Chirciu

Solution(Tran Hong)

For $x, y > 0$ we have $4(x^6 + y^6) \leq (3x^2 - 4xy + 3y^2)^3 \Leftrightarrow$

$$23x^6 - 108x^5y + 225x^4y^2 - 280x^3y^3 + 225x^2y^4 - 108xy^5 + 23y^6 \geq 0$$

$(x - y)^4(23x^2 - 16xy + 23y^2) \geq 0$ which result from $(x - y)^4 \geq 0$ equality for $x = y$

and $23x^2 - 16xy + 23y^2 > 0$ true from $\Delta = 16^2 - 4 \cdot 23^2 < 0$.

For $x = a, y = 1$

we get $4(a^6 + 1) \leq (3a^2 - 4a + 3)^3$ then $\sqrt[3]{4(b^6 + 1)} \leq 3a^2 - 4a + 3$

Equality for $a = 1$.

Similarly: $4(b^6 + 1) \leq (3b^2 - 4b + 3)^3$ and $4(c^6 + 1) \leq (3c^2 - 4c + 3)^3 \Rightarrow$

$$\Omega = \sum_{cyc} \frac{a^2}{2(2\mu - 1)b + \sqrt[3]{4(1 + b^6)}} \geq \sum_{cyc} \frac{a^2}{3b^2 - 4b + 3 + (4\mu - 2)b} =$$

$$= \sum_{cyc} \frac{a^2}{3b^2 + (4\mu - 6)b + 3} = \sum_{cyc} \frac{(a^2)^2}{3a^2b^2 + (4\mu - 6)a^2b + 3a^2} \overset{Bergstrom}{\geq}$$

$$\geq \frac{(a^2 + b^2 + c^2)^2}{3(a^2b^2 + b^2c^2 + c^2a^2 + a^2 + b^2 + c^2) + (4\mu - 6)(ba^2 + cb^2 + ac^2)} \overset{(*)}{\geq} \frac{3}{4\mu}$$

$$(*) \Leftrightarrow 4\mu\left(\sum a^4 + 2\sum a^2b^2\right) \geq$$

$$\geq 3\left(3\sum a^2b^2 + 3\sum a^2 + (4\mu - 6)(ba^2 + cb^2 + ac^2)\right)$$

$$4\mu\sum a^4 + 8\mu\sum a^2b^2$$

$$\geq 9\sum a^2b^2 + 9\sum a^2 + 3(4\mu - 6)(ba^2 + cb^2 + ac^2)$$

$$4\mu\sum a^4 + (8\mu - 9)\sum a^2b^2 \geq 3\sum a^2 + 3(4\mu - 6)(ba^2 + cb^2 + ac^2)$$

$$(4\mu - 6)\sum a^4 + (8\mu - 12)\sum a^2b^2 + \left[4\sum a^4 + 3\sum a^2b^2\right]$$

$$\geq 3\sum a^2 + 3(4\mu - 6)(ba^2 + cb^2 + ac^2)$$

$$(4\mu-6)\left(\sum a^4+2\sum a^2b^2-3(ba^2+cb^2+ac^2)\right)$$
$$+3\left(2\sum a^4+\sum a^2b^2-3\sum a^2\right)\overset{(**)}{\geq}0$$

With $a+b+c=3, \mu\geq\frac{3}{2}\Rightarrow 4\mu-6\geq 0$ we have:

$$\sum a^4+2\sum a^2b^2-3(ba^2+cb^2+ac^2)=$$
$$=\sum a^4+2\sum a^2b^2-(a+b+c)(ba^2+cb^2+ac^2)$$
$$=\sum a^4+2\sum a^2b^2-$$
$$-\left[\sum a^2b^2+abc(a+b+c)+(ba^3+cb^3+ac^3)\right]=$$

$$=\sum a^4+\sum a^2b^2-[abc(a+b+c)+(ba^3+cb^3+ac^3)]=$$

$$=\left[\sum a^4-(ba^3+cb^3+ac^3)\right]+\left[\sum a^2b^2-abc(a+b+c)\right]\overset{(1)}{\geq}0$$

We have (1) is true, because:

$$a^4+a^4+a^4+b^4\overset{AM-GM}{\geq}4\sqrt[4]{(a^3b)^4}=4ba^3$$
$$b^4+b^4+b^4+c^4\overset{AM-GM}{\geq}4\sqrt[4]{(b^3c)^4}=4cb^3$$
$$c^4+c^4+c^4+a^4\overset{AM-GM}{\geq}4\sqrt[4]{(c^3a)^4}=4ac^3$$

$$\Rightarrow 4(a^4+b^4+c^4)\geq 4(ba^3+cb^3+ac^3)$$
$$\Rightarrow\sum a^4-(ba^3+cb^3+ac^3)\overset{(2)}{\geq}0$$

$$a^2b^2+b^2c^2+c^2a^2\overset{AM-GM}{\geq}abc(a+b+c)$$
$$\Rightarrow\sum a^2b^2-abc(a+b+c)\overset{(3)}{\geq}0$$

From (1),(2)⇒(1) is true.

$$3\sum a^2 \overset{\sum a=3}{=} \frac{(\sum a)^2}{3}\cdot\sum a^2 \overset{BCS}{\leq} \sum a^2\cdot\sum a^2$$
$$=\left(\sum a^2\right)^2 \overset{(4)}{\leq} 2\sum a^4+\sum a^2b^2$$

$$(4)\Leftrightarrow \sum a^4 \geq \sum a^2b^2\ (true\ by\ x^2+y^2+z^2\geq xy+yz+zx)$$
$$2\sum a^4+\sum a^2b^2 \geq 3\sum a^2 \Rightarrow$$

$$3\left(2\sum a^4+\sum a^2b^2-3\sum a^2\right) \overset{(5)}{\geq} 0$$

From (4),(5) and $4\mu-6\geq 0\Rightarrow (**)$true $\Rightarrow (*)$true.

6.167 If $x,y,z>0, xy+yz+zx=3, \mu\geq\frac{13}{27}$ then:

$$\mu(x^2+y^2+z^2)+x^2y^2z^2\geq 3\mu+1$$

Marin Chirciu

Solution(Tran Hong)

$$x,y,z>0, xy+yz+zx=3 \Leftrightarrow \frac{x}{\sqrt{3}}\cdot\frac{y}{\sqrt{3}}+\frac{y}{\sqrt{3}}\cdot\frac{z}{\sqrt{3}}+\frac{z}{\sqrt{3}}\cdot\frac{x}{\sqrt{3}}=1$$

$\Rightarrow (\exists)\Delta ABC$ such that:

$$\frac{x}{\sqrt{3}}=tan\frac{A}{2};\frac{y}{\sqrt{3}}=tan\frac{B}{2};\frac{z}{\sqrt{3}}=tan\frac{C}{2}\Rightarrow x=\sqrt{3}tan\frac{A}{2};y=\sqrt{3}tan\frac{B}{2};$$
$$z=\sqrt{3}tan\frac{C}{2}$$

Hence, $\mu(x^2+y^2+z^2)+x^2y^2z^2\geq 3\mu+1;\ \mu\geq\frac{13}{27}\Leftrightarrow$

$$3\mu\sum_{cyc}tan^2\frac{A}{2}+27\left(\prod_{cyc}tan\frac{A}{2}\right)^2\geq 3\mu+1\Leftrightarrow$$

$$3\mu\cdot\frac{(4R+r)^2-2s^2}{s^2}+27\left(\frac{r}{s}\right)^2\geq 3\mu+1\Leftrightarrow$$

$$3\mu[(4R+r)^2-2s^2]+27r^2\geq(3\mu+1)s^2\Leftrightarrow$$

$$3\mu(4R+r)^2+27r^2\overset{(*)}{\geq}(9\mu+1)s^2$$

But: $s^2\leq 2R^2+10Rr-r^2+2(R-2r)\sqrt{R^2-2Rr}$

Let: $t=\frac{R}{r}\geq 2$, we must show that:

$$3\mu(4t+1)^2+27\geq(9\mu+1)\left[(2t^2+10t-1)+(2t-4)\sqrt{t^2-2t}\right]\Leftrightarrow$$

$$3\mu(4t+1)^2+27-(9\mu+1)(2t^2+10t-1)\geq(9\mu+1)(2t-4)\sqrt{t^2-2t}$$

$$3\mu(16t^2+8t+1)+27-(9\mu+1)(2t^2+10t-1)$$
$$\geq(9\mu+1)(2t-4)\sqrt{t^2-2t}\Leftrightarrow$$

$$2(t-2)[(15\mu-1)t-3\mu-7]\geq 2(9\mu+1)(t-2)\sqrt{t^2-2t}\Leftrightarrow$$

$$(t-2)[(15\mu-1)t-3\mu-7]\geq(9\mu+1)(t-2)\sqrt{t^2-2t}$$

Because: $t\geq 2\Rightarrow t-2\geq 0$

We need to prove:

$$(15\mu-1)t-3\mu-7\geq(9\mu+1)(t-2)\sqrt{t^2-2t}\Leftrightarrow$$

$$[(15\mu-1)t-3\mu-7]^2\geq(9\mu+1)^2(t^2-2t)$$

$$(15\mu-1)t-3\mu-7\overset{t\geq 2}{\geq}(15\mu-1)\cdot 2-3\mu-7=27\mu-9\overset{\mu\geq\frac{13}{27}}{\geq}13-9=4$$
$$>0\Leftrightarrow$$

$$(144\mu^2-48\mu)t^2+(72\mu^2-168\mu+16)t+9\mu^2+49\overset{(**)}{\geq}0\Leftrightarrow$$

$$(144\mu^2-48\mu)t^2+8(9\mu^2-21\mu+2)t+9\mu^2+49\overset{(**)}{\geq}0$$

Which is clearly true, because:

$$\Delta_t=-64(9\mu-4)(9\mu+1)^2\leq-\frac{16384}{27}<0;\left(\text{for }\mu\geq\frac{13}{27}\right)$$

$$a = 144\mu^2 - 48\mu = 48\pi(3\mu - 1) \overset{\mu \ge \frac{13}{27}}{\ge} 0 \Rightarrow (**)\text{true} \Rightarrow (*)\text{true}.$$

6.168 If $a, b, c > 0$; $a + b + c = 1$ then prove:

$$\frac{9}{136} \le \frac{a^2}{a^3+5} + \frac{b^2}{b^3+5} + \frac{c^2}{c^3+5} \le \frac{1}{6}$$

Jalil Hajimir

Solution(Khanh Hung Vu)

1) Prove that the inequality:

$$\frac{a^2}{a^3+5} + \frac{b^2}{b^3+5} + \frac{c^2}{c^3+5} \ge \frac{9}{136}; (1)$$

Put: $f(x) = \frac{x^2}{x^3+5}; x \in (0,1); f'(x) = \frac{x(10-x^3)}{(x^3+5)^2} > 0$ since

$$\begin{cases} x(10 - x^3) > 0 \\ (x^3 + 5)^2 > 0 \end{cases}; \big(x \in (0,1)\big) \Rightarrow f \text{ —increasing.}$$

Case 1. One of three number a, b, c isn't less than $\sqrt[3]{\frac{5}{2}(7 - 3\sqrt{5})}$. WLOG

$a \ge \sqrt[3]{\frac{5}{2}(7 - 3\sqrt{5})}$ and since f —increasing result:

$$f(a) \ge f\left(\sqrt[3]{\frac{5}{2}(7 - 3\sqrt{5})}\right) > \frac{9}{136}$$

On the other hand, we have $f(b) > 0; f(c) > 0 \Rightarrow f(a) + f(b) + f(c) > \frac{9}{136}$ which

$$\frac{a^2}{a^3+5} + \frac{b^2}{b^3+5} + \frac{c^2}{c^3+5} \ge \frac{9}{136}$$

Case 2. Both three number a, b, c less than $\sqrt[3]{\frac{5}{2}(7 - 3\sqrt{5})}$, which we have

$$a < \sqrt[3]{\frac{5}{2}(7-3\sqrt{5})}; b < \sqrt[3]{\frac{5}{2}(7-3\sqrt{5})}; c < \sqrt[3]{\frac{5}{2}(7-3\sqrt{5})}$$

We have:

$$f''(x) = \frac{2(x^6-35x^3+25)}{(x^3+5)^3} = \frac{2\left(x^3-\frac{5}{2}(7-3\sqrt{5})\right)\left(x^3-\frac{5}{2}(7+3\sqrt{5})\right)}{(x^3+5)^3} > 0$$

$$\forall x \in \left(0, \sqrt[3]{\frac{5}{2}(7-3\sqrt{5})}\right) \text{ since } \begin{cases} x^3-\frac{5}{2}(7-3\sqrt{5}) < 0 \\ x^3-\frac{5}{2}(7+3\sqrt{5}) < 0 \end{cases}$$

So f −convex, by Jensen Inequality, we have:

$$f(a)+f(b)+f(c) \geq f\left(\frac{a+b+c}{3}\right) = 3f\left(\frac{1}{3}\right)$$

$$\frac{a^2}{a^3+5}+\frac{b^2}{b^3+5}+\frac{c^2}{c^3+5} \geq \frac{9}{136}$$

The inequality occurs when $a=b=c=\frac{1}{3}$

2) Prove the inequality:

$$\frac{a^2}{a^3+5}+\frac{b^2}{b^3+5}+\frac{c^2}{c^3+5} \leq \frac{1}{6}; (2)$$

By AM-GM inequality, we have:

$$x^3+5 = x^3+1+1+3 \geq 3x+3; \forall x > 0$$

On the other hand, we have: $(x-1)(2x+1) < 0 \Rightarrow$

$$2x^2-x-1 < 0 \Rightarrow x+1 > 2x^2; \forall x < 1$$

So, $x^3+5 > 6x^2 \Leftrightarrow \frac{x^2}{x^3+5} < \frac{x}{6}; \forall x \in (0,1) \Rightarrow$

$$\frac{a^2}{a^3+5}+\frac{b^2}{b^3+5}+\frac{c^2}{c^3+5} < \frac{a+b+c}{6} \Rightarrow \frac{a^2}{a^3+5}+\frac{b^2}{b^3+5}+\frac{c^2}{c^3+5} \leq \frac{1}{6}$$

From (1),(2) we have the thing to prove.

6.169 If $x > 1, p, q, r \in \mathbb{N}$ then:

$$\frac{(x+1)^{2(p+q+r)}(x^2-1)^3}{(x^{2p+2}-1)(x^{2q+2}-1)(x^{2r+2}-1)} \le \frac{(2p)!\,(2q)!\,(2r)!}{p!\,q!\,r!}$$

Daniel Sitaru

Solution(Ravi Prakash)

$$\text{Let } f(x) = \frac{(x+1)^{2m}}{x^{2m}+x^{2m-2}+\dots+x^2+1}, m \in \mathbb{N}, x \ge 1$$

$$\text{For } x > 1, f'(x) =$$

$$= \frac{(x^{2m}+x^{2m-2}+\dots+x^2+1)(2m)(x+1)^{2m-1}-(x+1)^{2m}[2mx^{2m-1}+\dots+2x]}{(x^{2m}+x^{2m-2}+\dots+x^2+1)^2}$$

Numerator of $f'(x)$ is $2(x+1)^{2m-1}p(x)$ when

$$p(x) = m(x^{2m}+x^{2m-2}+\dots+x^2+1)-(x+1)(mx^{2m-1}+\dots+x) =$$

$$= x^{2m-2}+2x^{2m-4}+\dots+(m-1)x^2+m-mx^{2m-1}-(m-1)x^{2m-3}-\dots-x =$$

$$= (x^{2m-2}+x^{2m-4}+\dots+x^2+1-mx^{2m-1})+(x^{2m-4}+x^{2m-6}+\dots.+x^2+1-$$
$$-(m-2)x^{2m-5}+\dots+(1-x) < 0, \forall x > 1 \Rightarrow f(x) < \frac{2^{2m}}{m+1}$$

Now, we show that:

$$\frac{2^{2m}}{m+1} \le \frac{(2m)!}{m!} = (2m)(2m-1)\dots(m+1), \forall m \in \mathbb{N}$$

For $m = 1, Lhs = 2, Rhs = 2$.

For $m = 2, Lhs = \frac{16}{3}, Rhs = 4 \cdot 3 \Rightarrow Lhs \le Rhs$

For $m \ge 3, (m+1)(m+2)\dots(2m) \ge \underbrace{4 \cdot 4 \cdot \dots \cdot 4}_{m-times} = 4^m = 2^{2m} > \frac{2^{2m}}{m+1}$

Thus, $f(x) < \frac{2^{2m}}{m+1}, \forall m \in \mathbb{N}, \forall x \ge 1$.

$$\frac{(x+1)^{2m}}{x^{2m}+x^{2m-2}+\cdots+x^2+1}\le\frac{(2m)!}{m!},\forall x>1\Rightarrow$$

$$\frac{(x^2-1)(x+1)^{2m}}{(x^2-1)(x^{2m}+x^{2m-2}+\cdots+x^2+1)}\le\frac{(2m)!}{m!},\forall x>1$$

$$\frac{(x+1)^{2m}(x^2-1)}{x^{2m+2}-1}\le\frac{(2m)!}{m!},\forall x>1$$

Taking $m=p,q,r$ and multiplying, we get

$$\frac{(x+1)^{2p+2q+2r}(x^2-1)^3}{(x^{2p+2}-1)(x^{2q+2}-1)(x^{2r+2}-1)}\le\frac{(2p)!\,(2q)!\,(2r)!}{p!\,q!\,r!}$$

6.170 If $0<a\le b\le\frac{\sqrt{3}}{3}$ then:

$$\boldsymbol{a^2b^2(2-a-b)^2(2+a+b)^2\le(1-a^2)(1-b^2)(a+b)^4}$$

Daniel Sitaru

Solution(Remus Florin Stanca)

$$a^2b^2(2-a-b)^2(2+a+b)^2\le(1-a^2)(1-b^2)(a+b)^4\Leftrightarrow$$

$$\frac{a}{1-a^2}\cdot\frac{b}{1-b^2}\cdot\frac{ab}{(a+b)^2}\le\left(\frac{a+b}{4-(a+b)^2}\right)^2\Leftrightarrow$$

$$\frac{1}{4}\cdot\frac{2a}{1-a^2}\cdot\frac{2b}{1-b^2}\cdot\frac{ab}{(a+b)^2}\le\left(\frac{\frac{a+b}{4}}{1-\left(\frac{a+b}{2}\right)^2}\right)^2\Leftrightarrow$$

$$\frac{2a}{1-a^2}\cdot\frac{2b}{1-b^2}\cdot\frac{ab}{(a+b)^2}\le\left(\frac{2\cdot\frac{a+b}{2}}{1-\left(\frac{a+b}{2}\right)^2}\right)^2\cdot\frac{1}{4}\Leftrightarrow$$

Let: $tan\frac{x}{2}=a,tan\frac{y}{2}=b,\frac{tan\frac{x}{2}+tan\frac{y}{2}}{2}=tan\alpha\Leftrightarrow$

$$tanxtany\cdot\frac{tan\frac{x}{2}tan\frac{y}{2}}{tan^2\alpha}\le tan^2(2\alpha)\Leftrightarrow$$

$$tanxtany\cdot tan\frac{x}{2}tan\frac{y}{2}\le tan^2\alpha tan^2(2\alpha);(1)$$

Let's prove that: $tan\frac{x}{2}tan\frac{y}{2}\le tan^2\alpha\Leftrightarrow tan\frac{x}{2}tan\frac{y}{2}\le\left(\frac{tan\frac{x}{2}+tan\frac{y}{2}}{2}\right)^2$

$$(true\ by\ \frac{a+b}{2}\ge\sqrt{ab},\forall a,b\ge 0;(2)$$

Let's prove that: $tanxtany\le tan^2(2\alpha)\Leftrightarrow$

$$\frac{1}{2}log\left(\frac{a}{1-a^2}\right)+\frac{1}{2}log\left(\frac{b}{1-b^2}\right)\le log\left(\frac{\frac{a+b}{2}}{1-\left(\frac{a+b}{2}\right)^2}\right)$$

Let: $f(x)=log\left(\frac{x}{1-x^2}\right)\Rightarrow\frac{\partial f}{\partial x}=\frac{1}{2}\cdot\frac{1-x^2}{x}\cdot\left(\frac{1}{1-x}-\frac{1}{1+x}\right)'=$

$$\frac{1}{2}\cdot\frac{1-x^2}{x}\cdot\left(\frac{1}{(x-1)^2}+\frac{1}{(x+1)^2}\right)=\frac{x^2+1}{x-x^3}$$

$$\Rightarrow\frac{\partial^2 f}{\partial x^2}=\frac{(3x^2-1)(x^2+1)}{(x-x^3)^2}\le 0\left(a\le\frac{1}{\sqrt{3}}\right)\Rightarrow f-concave\Rightarrow$$

$$\frac{1}{2}log\left(\frac{a}{1-a^2}\right)+\frac{1}{2}log\left(\frac{b}{1-b^2}\right)\le log\left(\frac{\frac{a+b}{2}}{1-\left(\frac{a+b}{2}\right)^2}\right)\Rightarrow true(3)$$

$$(2);(3)\Rightarrow(1)true\Rightarrow a^2b^2(2-a-b)^2(2+a+b)^2$$

$$\le(1-a^2)(1-b^2)(a+b)^4$$

6.171 If $x, y, z \geq 0$ then:

$$\left(\prod_{cyc}(x+1)+\prod_{cyc}(2x+1)\right)\prod_{cyc}(x+2) \geq 2\prod_{cyc}(3x+2)$$

Daniel Sitaru

Solution(George Florin Serban)

$$\prod_{cyc}(x+1)+\prod_{cyc}(2x+1)$$

$$\overset{AM-GM}{\geq} 2\sqrt{\prod_{cyc}(x+1)(2x+1)} \overset{(1)}{\geq} \frac{2\prod_{cyc}(3x+2)}{\prod_{cyc}(x+2)}$$

$$(1) \Leftrightarrow \prod_{cyc}(x+1)(2x+1) \geq \left(\frac{\prod_{cyc}(3x+2)}{\prod_{cyc}(x+2)}\right)^2 \Leftrightarrow$$

$$\prod_{cyc}(x+1)(2x+1)(x+2)^2 \geq \prod_{cyc}(3x+2)^2 \Leftrightarrow$$

$$(2x^2+3x+1)(x^2+4x+4) \geq 9x^2+12x+4 \Leftrightarrow$$

$$2x^4+11x^3+12x^2+4x \geq 0, \forall x \geq 0\ (true).$$

REFFERENCES

1.Romanian Mathematical Magazine-Interactive Journal-www.ssmrmh.ro

2.Mihaly Bencze,Daniel Sitaru,Marian Ursărescu:"*Olympic Mathematical Energy***"*-Studis-Publishing House*-Iași-2018**

3.Daniel Sitaru, George Apostolopoulos:"*The Olympic Mathematical Marathon***"-Publishing House- Cartea Românească-Pitești-2018**

4. Mihaly Bencze,Daniel Sitaru:"*Quantum Mathematical Power***" -Publishing House Studis, Iasi-2018**

5.Daniel Sitaru,Marian Ursărescu:"*Calculus Marathon***"*-Studis-Publishing House*-Iasi-2018**

6.Daniel Sitaru, Mihaly Bencze:"*699 Olympic Mathematical Challenges***"-Publishing House Studis, Iasi-2017**

7.Daniel Sitaru,Marian Ursărescu:"*Ice Math-Contests Problems***"*-Studis-Publishing House*-Iasi-2019**