5043

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5043. Let $b \geq a \geq 1$. Prove that

$$\int_a^b \int_a^b \frac{dxdy}{1+\sqrt{xy}} \leq (b-a)\ln\Big(\frac{b+1}{a+1}\Big).$$

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Solution 1 by editors.
Since for any $a, b \geq 1$ we have

$$2(b-a)\ln\Big(\frac{b+1}{a+1}\Big) = \int_a^b \frac{dx}{1+x} \cdot \int_a^b dy + \int_a^b dx \cdot \int_a^b \frac{dy}{1+y}$$
$$= \int_a^b \int_a^b \Big(\frac{1}{1+x} + \frac{1}{1+y}\Big) dxdy,$$

it suffices to show that

$$\frac{2}{1+\sqrt{xy}} \leq \frac{1}{1+x} + \frac{1}{1+y}, \qquad x, y \geq 1.$$

The latter inequality is readily seen to be equivalent to

$$(\sqrt{xy}-1)(\sqrt{x}-\sqrt{y})^2 \geq 0, \qquad x, y \geq 1$$

which is trivial. This concludes the proof. □

Solution 2 by proposer.

$$\int_a^b \int_a^b \frac{dxdy}{1+\sqrt{xy}} \leq (b-a)\ln\Big(\frac{b+1}{a+1}\Big)$$
$$2\int_a^b \int_a^b \frac{dxdy}{1+\sqrt{xy}} \leq 2(b-a)\ln\Big(\frac{b+1}{a+1}\Big) =$$
$$= (b-a)\ln\Big(\frac{b+1}{a+1}\Big) + (b-a)\ln\Big(\frac{b+1}{a+1}\Big) =$$
$$= \int_a^b \frac{1}{1+x} dx \cdot \int_a^b dy + \int_a^b dx \cdot \int_a^b \frac{1}{1+y} dy =$$
$$= \int_a^b \int_a^b \Big(\frac{1}{1+x} + \frac{1}{1+y}\Big) dxdy$$
$$\int_a^b \int_a^b \frac{2}{1+\sqrt{xy}} dxdy \leq \int_a^b \int_a^b \Big(\frac{1}{1+x} + \frac{1}{1+y}\Big) dxdy$$

Remains to prove that:

$$\frac{2}{1+\sqrt{xy}} \leq \frac{1}{1+x} + \frac{1}{1+y}; (\forall) x, y \geq 1$$

$$\frac{2}{1+\sqrt{xy}} \le \frac{1+x+1+y}{(1+x)(1+y)}$$
$$(2+x+y)(1+\sqrt{xy}) \ge 2(1+x)(1+y)$$
$$2+2\sqrt{xy}+(x+y)+(x+y)\sqrt{xy} \ge 2(1+x+y+xy)$$
$$2+2\sqrt{xy}+(x+y)+(x+y)\sqrt{xy} \ge 2+2(x+y)+2xy$$
$$\sqrt{xy}(2+x+y) \ge x+y+2xy$$
$$(x+y)\sqrt{xy}+2\sqrt{xy}-(x+y)-2xy \ge 0$$
$$(x+y)(\sqrt{xy}-1)-2\sqrt{xy}(\sqrt{xy}-1) \ge 0$$
$$(\sqrt{xy}-1)(x+y-2\sqrt{xy}) \ge 0$$
$$(\sqrt{xy}-1)(\sqrt{x}-\sqrt{y})^2 \ge 0$$

True because $x, y \ge 1 \Rightarrow \sqrt{xy} \ge 1 \Rightarrow \sqrt{xy}-1 \ge 0$.
Equality holds for $a = b$. □

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