ROMANIAN MATHEMATICAL MAGAZINE

**Let $a,b,c\geq 0, a+b+c=4$. Prove that :**

$$\sqrt{9b-bc+9c}+\sqrt{9c-ca+9a}+\sqrt{9a-ab+9b}\leq 10\sqrt{2}$$

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Let $p:=a+b+c=4,\ \ q:=ab+bc+ca\leq \frac{p^2}{3}=\frac{16}{3},\ \ r:=abc.$

By CBS inequality, we have

$$\sum_{cyc}\sqrt{9b-bc+9c}\leq\sqrt{\sum_{cyc}(9b-bc+9c)(a+6).\sum_{cyc}\frac{1}{a+6}}=$$

$$=\sqrt{(108p+12q-3r).\frac{12p+q+108}{216+36p+6q+r}}\overset{?}{\leq}10\sqrt{2}$$

$$\overset{p=4}{\Leftrightarrow}\ 4608-1104q-12q^2+(668+3q)r\geq 0.\ (*)$$

If $q\leq 4:\ LHS_{(*)}\geq 4608-1104q-12q^2=(4-q)(1152+12q)\geq 0.$

If $4\leq\ q\leq\frac{16}{3}:\ LHS_{(*)}\overset{Schur}{\geq}\ 4608-1104q-12q^2+\frac{(668+3q)(4pq-p^3)}{9}=$

$$=\frac{20(q-4)(16-3q)}{9}\geq 0.$$

Equality holds iff $a=b=c=\frac{4}{3}$ & $a=b=2,c=0$ and its permutation.