ROMANIAN MATHEMATICAL MAGAZINE

Let $a,b,c\geq 0, ab+bc+ca+abc=4$. Prove that :

$$\sqrt{a+ab+b}+\sqrt{b+bc+c}+\sqrt{c+ca+a}\leq\sqrt{5a+5b+5c+12}$$

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By CBS inequality, we have

$$\sum_{cyc}\sqrt{b+bc+c}\leq\sqrt{\sum_{cyc}(b+bc+c)(a+2).\sum_{cyc}\frac{1}{a+2}}$$

From the given condition, we have

$$\frac{1}{a+2}+\frac{1}{b+2}+\frac{1}{c+2}=1.$$

So it suffices to prove that

$$\sum_{cyc}(b+bc+c)(a+2)\leq 5a+5b+5c+12 \text{ or } ab+bc+ca\leq a+b+c.$$

Let $p:=a+b+c, q:=ab+bc+ca, r:=abc$. We have $4=q+r\leq\frac{p^2}{3}+\frac{p^3}{27}$, then $p\geq 3$.

If $p\geq 4$: $ab+bc+ca\leq 4\leq a+b+c$.

If $3\leq p\leq 4$: $4pq\overset{Schur}{\leq} p^3+9r=p^3+9(4-q)\Rightarrow q\leq\frac{p^3+36}{4p+9}=p-\frac{(p^2-9)(4-p)}{4p+9}\leq p$.

Equality holds if $a=b=c=1$ & $a=b=2, c=0$ and permutations.