ROMANIAN MATHEMATICAL MAGAZINE

Let $a,b,c\geq 0, a+b+c=ab+bc+ca>0$. Prove that :

$$\frac{a(3a+5bc)}{b+c}+\frac{b(3b+5ca)}{c+a}+\frac{c(3c+5ab)}{a+b}\geq 12$$

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Solution by Mohamed Amine Ben Ajiba-Tanger-Morocco

Let $p:=a+b+c,\ \ q:=ab+bc+ca,\ \ r:=abc$. We have $p=q\leq\frac{p^2}{3}$, then $p\geq 3$.

By AM − GM inequality, we have

$$\frac{a^2}{b+c}+\frac{a^2(b+c)}{4}\geq a^2\ \ \text{(and analogs)}$$

$$\Rightarrow \sum_{cyc}\frac{a^2}{b+c}\geq a^2+b^2+c^2-\frac{1}{4}\sum_{cyc}a^2(b+c)=p^2-2q-\frac{1}{4}(pq-3r)\overset{q=p}{=}\frac{3p^2}{4}-2p+\frac{3r}{4}.\quad (1)$$

Then

$$\sum_{cyc}\frac{a(3a+5bc)}{b+c}\overset{(1)\ \&\ CBS}{\geq} 3\left(\frac{3p^2}{4}-2p+\frac{3r}{4}\right)+5abc.\frac{9}{2(a+b+c)}=\frac{9p^2}{4}-6p+\frac{9}{4}\left(1+\frac{10}{p}\right)r$$
$$=f(r).$$

If $p\geq 4$, we have $f(r)\geq f(0)=12+\frac{(p-4)(9p+12)}{4}\geq 12$.

If $3\leq p\leq 4$, we have $f(r)\overset{Schur}{\geq} f\left(\frac{4pq-p^3}{9}\right)\overset{q=p}{=} 12+\frac{(p-3)(4-p)(p+4)}{4}\geq 12$.

Equality holds iff $a=b=c=1$ & $a=b=2, c=0$ and its permutations.