ROMANIAN MATHEMATICAL MAGAZINE

Let $a,b,c\geq 0$ such that $a+b+c=5$ and $ab+bc+ca>0$. Prove that :

$$\frac{a\sqrt{a+bc}}{b+c}+\frac{b\sqrt{b+ca}}{c+a}+\frac{c\sqrt{c+ab}}{a+b}\geq\sqrt{10}$$

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By $AM-GM$ inequality, we have

$$\sum_{cyc}\frac{a\sqrt{a+bc}}{b+c}=2\sqrt{10}.\sum_{cyc}\frac{a(a+bc)}{2.\sqrt{2}(b+c).\sqrt{5(a+bc)}}\geq 2\sqrt{10}.\sum_{cyc}\frac{a(a+bc)}{2(b+c)^2+5(a+bc)}$$

$$=\sqrt{10}+\sqrt{10}.\sum_{cyc}a.\frac{5(a+bc)-2(b+c)^2}{5[2(b+c)^2+5(a+bc)]}=\sqrt{10}+\frac{\sqrt{10}}{5}.M.$$

So it suffices to prove that $M\geq 0$. WLOG, we assume that $a\geq b\geq c$.

Let $A=2(b+c)^2+5(a+bc), B=2(c+a)^2+5(b+ca), C=2(a+b)^2+5(c+ab)$.

Since $C-B=(b-c)(5+7a)\geq 0$ and $B-A=(a-b)(5+7c)\geq 0$, then we have $A\leq B\leq C$,

and since $5(c+ab)-2(a+b)^2=(a+b+c)c+ab-2a^2-2b^2\leq 0$, and

$$5(a+bc)-2(b+c)^2=(a+b+c)a+bc-2b^2-2c^2=(a-b)(a+2b)+c(a+b-c)\geq 0,$$

Then

$$M=\frac{a[5(a+bc)-2(b+c)^2]}{A}+\frac{b[5(b+ca)-2(c+a)^2]}{B}+\frac{c[5(c+ab)-2(a+b)^2]}{C}$$

$$\overset{A\geq B\geq C}{\geq}\frac{a[5(a+bc)-2(b+c)^2]}{B}+\frac{b[5(b+ca)-2(c+a)^2]}{B}+\frac{c[5(c+ab)-2(a+b)^2]}{B}$$

$$=\frac{a^3+b^3+c^3+3abc-a^2(b+c)-b^2(c+a)-c^2(a+b)}{B}\overset{Schur}{\geq}0.$$

Equality holds iff $\left(a=b=c=\frac{5}{3}\right)$ and $\left(a=b=\frac{5}{2},c=0\right)$ and its permutations.