ROMANIAN MATHEMATICAL MAGAZINE

Let $a,b,c\geq 0, ab+bc+ca>0$, then prove that :

$$\sqrt{\frac{a(b+c)}{(2b+c)(2c+b)}}+\sqrt{\frac{b(c+a)}{(2c+a)(2a+c)}}+\sqrt{\frac{c(a+b)}{(2a+b)(2b+a)}}\geq\sqrt{2}$$

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Solution by Mohamed Amine Ben Ajiba-Tanger-Morocco

Let $a=x^2,b=y^2,c=z^2$, where $x,y,z\geq 0$. By GM − HM inequality, we have

$$\frac{(2b+c)(2c+b)}{b+c}=2(b+c)+\frac{bc}{b+c}\leq 2(b+c)+\frac{\sqrt{bc}}{2}=\frac{4(y^2+z^2)+yz}{2}\ \text{(and analogs)}.$$

Then

$$\sum_{cyc}\sqrt{\frac{a(b+c)}{(2b+c)(2c+b)}}\geq\sqrt{2}\sum_{cyc}\frac{x}{\sqrt{4(y^2+z^2)+yz}}\overset{Hölder}{\geqslant}\sqrt{\frac{2(x+y+z)^3}{\sum_{cyc}x(4y^2+yz+4z^2)}}\overset{?}{\geqslant}\sqrt{2}$$

$$\Leftrightarrow x^3+y^3+z^3+3xyz\geq x^2(y+z)+y^2(z+x)+z^2(x+y),$$

which is Schur's inequality. So the proof is complete.

Equality holds iff $(a=b=c)$ and $(a=b>0,c=0)$ and its permmutations.