ROMANIAN MATHEMATICAL MAGAZINE

Let $a,b,c\geq 0, a+b+c=3$. Prove that :

$$\frac{ab-1}{2a+2b-11}+\frac{bc-1}{2b+2c-11}+\frac{ca-1}{2c+2a-11}\geq 0$$

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Let $p:=a+b+c=3, q:=ab+bc+ca, r:=abc$. The desired inequality is equivalent to

$$\sum_{cyc}\frac{bc-1}{2b+2c-11}\geq 0 \Leftrightarrow \sum_{cyc}\frac{1-bc}{11-2(3-a)}\geq 0 \Leftrightarrow \sum_{cyc}\frac{1-bc}{5+2a}\geq 0$$

$$\Leftrightarrow \sum_{cyc}(1-bc)(5+2b)(5+2c)\geq 0$$

$$\Leftrightarrow 75+20\sum_{cyc}a-21\sum_{cyc}bc-10\sum_{cyc}bc(b+c)-4\sum_{cyc}(bc)^2\geq 0$$

$$\Leftrightarrow 135-51q-4q^2+54r\geq 0. \quad (1)$$

Since $q\leq \frac{p^2}{3}=3$, then we have two cases:

If $q\leq \frac{9}{4}$, we have : $LHS_{(1)}\geq 135-51q-4q^2=(9-4q)(15+q)\geq 0$.

If $\frac{9}{4}\leq q\leq 3$, we have :

$$LHS_{(1)}\overset{Schur}{\geq} 135-51q-4q^2+6(12q-27)=(3-q)(4q-9)\geq 0.$$

Equality holds iff $a=b=c=1$ & $a=b=\frac{3}{2}, c=0$ and permutations.